The Sample Complexity of Revenue Maximization

in the Hierarchy of Deterministic Combinatorial Auctions

Ellen Vitercik

Joint work with Nina Balcan and Tuomas Sandholm

Theory Lunch

27 April 2016

Combinatorial (multi-item) auctions

: $5

: $5

: $6

Combinatorial auctions allow bidders to express preferences for bundles of goods

Real-world examples

- US Government wireless spectrum auctions [FCC]
- Sourcing auctions [Sandholm 2013]
- Airport time slot allocation [Rassenti 1982]
- Building development, e.g. office space in GHC (no money)
- Property sales

Mechanism design

- Mechanism designer must determine:
  - Allocation function: Who gets what?
  - Payment function: What does the auctioneer charge?
- Goal: design strategy-proof mechanisms
  - Easy for the bidders to compute the optimal strategy
  - Easy for designer to analyze possible outcomes

Warm-up: single-item auctions

NINA

: $5

, -$3

TUOMAS

: $3

ø, -$0

Second-Price Auction

Allocation (N:$5, T:$3)

= give carrot to Nina

Payment (N:$5, T:$3)

= charge Nina $3

Second-price auction: the classic strategy-proof, single-item auction.

Revenue-maximizing combinatorial auctions

- **Standard assumptions:** bidders' valuations drawn from distribution D, mechanism designer knows D
  - Allocation and payment rules often depend on D

Revenue-maximizing combinatorial auctions

Design Challenges	Feasible Solutions
Support of D might be doubly-exponential	Draw samples from D instead
NP-hard to determine the revenue-maximizing deterministic auction with respect to D [Conitzer and Sandholm 2002]	Fix a rich class of auctions. Can we learn the revenue-maximizing combinatorial auction in that class with respect to D given samples drawn from D?

- **Central problem in Automated Mechanism Design** [Conitzer and Sandholm 2002, 2003, 2004, Likhodedov and Sandholm 2004, 2005, 2015, Sandholm 2003]

No theory that relates the performance of the designed mechanism on the samples to that mechanism's expected performance on D, until now.

Outline

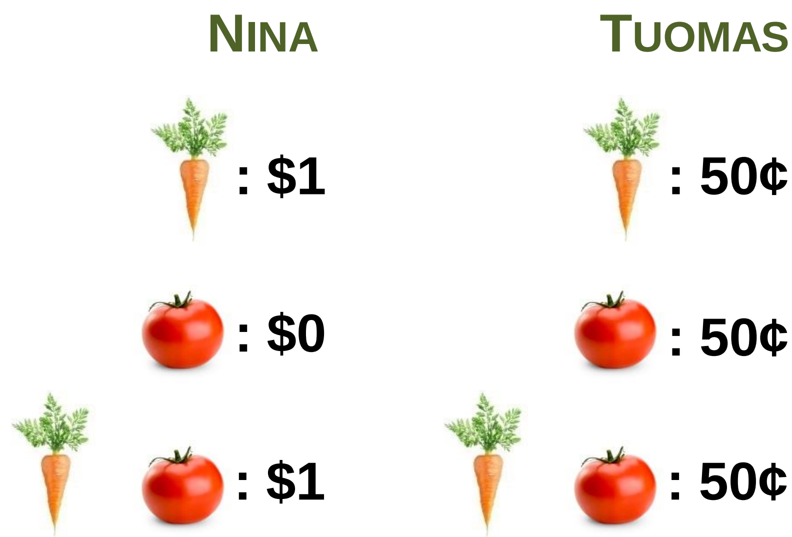
- Introduction
- Hierarchy of deterministic combinatorial auction classes
- Our contribution: how many samples are needed to learn over the hierarchy of auctions?
- Affine maximizer auctions and Rademacher complexity
- Mixed-bundling auctions and pseudo-dimension
- Summary and future directions

Combinatorial auctions

Nina	Tuomas
 : \$1	 : 50¢
 : \$0	 : 50¢
  : \$1	  : 50¢

- 3^2 possible outcomes $\boldsymbol{o} = (\boldsymbol{o_1}, \boldsymbol{o_2})$
- For example, $\boldsymbol{o} = ($\{  \}, \{  \}$)$

A natural generalization of second price

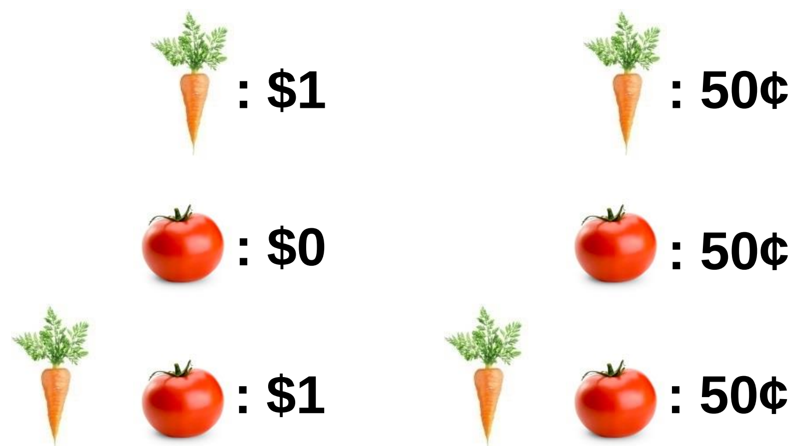


- Social Welfare (o)
  = SW(o) = $\sum_{i \in Bidders} v_i(o)$ — o^* maximizes SW(o)
- SW$_{-i}(o)$ = $\sum_{j \in Bidders - \{i\}} v_j(o)$ — o^{-i} maximizes SW$_{-i}(o)$
- Allocation: o^*
- Payment: Nina pays SW$_{-Nina}\left(o^{-Nina}\right)$ – SW$_{-Nina}(o^*)$

The “Vickrey-Clarke-Groves mechanism” (VCG).

VCG in action

NINA **TUOMAS**

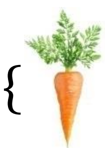
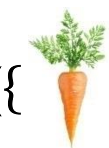


- $\boldsymbol{o}^* = (\{$  $\}, \{$  $\})$
- $\boldsymbol{o}^{-Nina} = (\emptyset, \{$  $,$  $\})$
- **Nina pays** $v_{Tuomas}(\{$  $,$  $\}) - v_{Tuomas}(\{$  $\})$ **= 0**

How do we get the bidders to pay more?

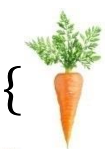
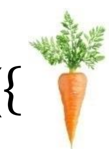
Outcome boosting

Nina	Tuomas
 : \$1	 : 50¢
 : \$0	 : 50¢
  : \$1	  : 50¢

- **value**$(\emptyset, \{$  ,  $\}) = v_{Nina}(\emptyset) + v_{Tuomas}$**(**$\{$  ,  $\}$**) = 50¢**

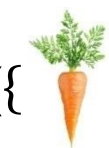
Outcome boosting

	Nina	Tuomas
	: $1	 : 50¢
	: $0	 : 50¢
 	: $1	  : 50¢

- **value**$(\emptyset,\ \{$  ,  $\}) = v_{Nina}(\emptyset) + v_{Tuomas}(\{$  ,  $\})$ **= 50¢ + 99¢**
- $o^* = (\{$  $\},\ \{$  $\})$
- $o^{-Nina} = (\emptyset,\ \{$  ,  $\})$

Outcome boosting

Nina	Tuomas
 : $1	 : 50¢
 : $0	 : 50¢
  : $1	  : 50¢

- value($\emptyset$, {  ,  }) $= v_{Nina}(\emptyset) + v_{Tuomas}$({  ,  }) = 50¢ + 99¢
- $o^* =$ ({  }, {  })
- $o^{-Nina} =$ ($\emptyset$, {  ,  })
- Nina pays v_{Tuomas}({  ,  }) + 99¢ $- v_{Tuomas}$({  }) = 99¢

Affine maximizer auctions (AMAs)

- **Boost outcomes:** $\lambda(o)$
- **Take bids** v
- **Compute outcome:**

$$o^* = argmax_o\{SW(o) + \lambda(o)\}$$

- **Compute Bidder** i**'s payment:**

$$SW_{-i}\left(o^{-i}\right) + \lambda\left(o^{-i}\right) - \left(SW_{-i}(o^*) + \lambda(o^*)\right)$$

Affine maximizer auctions (AMAs)

- **Boost outcomes:** $\boldsymbol{\lambda(o)}$
- **Take bids** $\boldsymbol{v}$
- **Compute outcome:**

$$\boldsymbol{o^* = argmax_{\,o} \left\{ \sum_{j \in Bidders}^{n} v_j(o) + \lambda(o) \right\}}$$

- **Compute Bidder** $\boldsymbol{i}$**'s payment:**

$$\boldsymbol{\left[\left(\sum_{j \in Bidders - \{i\}} v_j\left(o^{-i}\right) + \lambda\left(o^{-i}\right) \right) - \left(\sum_{j \in Bidders - \{i\}} v_j(o^*) + \lambda(o^*) \right) \right]}$$

Affine maximizer auctions (AMAs)

- **Boost outcomes: $\lambda(o)$; Weight bidders: w_i**
- **Take bids v**
- **Compute outcome:**

$$o^* = argmax_{\,o} \left\{ \sum_{j \in Bidders}^{n} v_j(o) + \lambda(o) \right\}$$

- **Compute Bidder i's payment:**

$$\left[\left(\sum_{j \in Bidders-\{i\}} v_j(o^{-i}) + \lambda(o^{-i}) \right) - \left(\sum_{j \in Bidders-\{i\}} v_j(o^*) + \lambda(o^*) \right) \right]$$

Affine maximizer auctions (AMAs)

- **Boost outcomes: $\lambda(o)$; Weight bidders: w_i**
- **Take bids v**
- **Compute outcome:**

$$o^* = argmax_{\,o}\left\{\sum_{j \in Bidders}^{n} w_j v_j(o) + \lambda(o)\right\}$$

- **Compute Bidder i's payment:**

$$\left[\left(\sum_{j \in Bidders-\{i\}} v_j\left(o^{-i}\right) + \lambda\left(o^{-i}\right)\right) - \left(\sum_{j \in Bidders-\{i\}} v_j\left(o^*\right) + \lambda\left(o^*\right)\right)\right]$$

Affine maximizer auctions (AMAs)

- **Boost outcomes:** $\lambda(o)$**; Weight bidders:** w_i
- **Take bids** v
- **Compute outcome:**

$$o^* = argmax_{\,o} \left\{ \sum_{j \in Bidders}^{n} w_j v_j(o) + \lambda(o) \right\}$$

- **Compute Bidder** i**'s payment:**

$$\frac{1}{w_i}\left[\left(\sum_{j \in Bidders-\{i\}} w_j v_j(o^{-i}) + \lambda(o^{-i})\right) - \left(\sum_{j \in Bidders-\{i\}} w_j v_j(o^*) + \lambda(o^*)\right)\right]$$

Hierarchy of parameterized auction classes

Affine maximizer auctions [R79]	$w_i, \lambda(o) \in \mathbb{R}$

$\cup$ $\cup$

Virtual valuation combinatorial auctions [SL03]	$\lambda(o) = \sum_{i \in Bidders} \lambda_i(o)$
λ-auctions [J07]	• $w_i = 1$ • $\lambda(o) \in \mathbb{R}$

$\cup$ $\cup$

Mixed bundling auctions with reserve prices [TS12]	• $w_i = 1$ • $\lambda(o) = 0$ except any outcome where a bidder gets all items • item reserve prices

$\cup$

Mixed bundling auctions [J07]	• $w_i = 1$ • $\lambda(o) = 0$ except outcome where a bidder gets all items

Outline

- Introduction
- Hierarchy of deterministic combinatorial auction classes
- Our contribution: how many samples are needed to learn over the hierarchy of auctions?
- Affine maximizer auctions and Rademacher complexity
- Mixed-bundling auctions and pseudo-dimension
- Summary and future directions

Our contribution

- **Optimize $\lambda(o)$ and w given a sample $S \sim D^N$**
  - **(Automated Mechanism Design)**
- **We want:**
  - **The auction with best revenue over the sample has almost optimal expected revenue**
  - **Any approximately revenue-maximizing auction over the sample will have approximately optimal expected revenue**
- **For any auction we output, we want $|S|$ large enough such that:**

  |empirical revenue – expected revenue| < ϵ
- **In other words, how many samples $|S| = N$ do we need to ensure that**

  |empirical revenue – expected revenue|

  $$= \left| \frac{1}{N} \sum_{v \in S} rev_A(v) - \mathbb{E}_{v \sim D}[rev_A(v)] \right| < \epsilon$$

  for all auctions A in the class?
- **(We can only do this with high probability.)**

How many samples do we need?

Affine maximizer auctions [R79]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

∪ ∪

Virtual valuation combinatorial auctions [SL03]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

λ-auctions [J07]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

∪ ∪

Mixed bundling auctions with reserve prices [TS12]

$$N = \tilde{O}\left((U/\epsilon)^2 m^3\right)$$

∪

Mixed bundling auctions [J07]

$$N = \tilde{O}\left((U/\epsilon)^2\right)$$

Variables

N: sample size
n: # bidders
m: # items
U: maximum revenue achievable over the support of the bidders' valuation distributions

How many samples do we need?

Affine maximizer auctions [R79]
$$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$$

∪ ∪

Virtual valuation combinatorial auctions [SL03]
$$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$$

λ-auctions [J07]
$$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$$

∪ ∪

Mixed bundling auctions with reserve prices [TS12]
$$N = \tilde{O}\left((U/\epsilon)^2 m^3\right)$$

∪

Mixed bundling auctions [J07]
$$N = \tilde{O}\left((U/\epsilon)^2\right)$$

Variables

N: sample size
n: # bidders
m: # items
U: maximum revenue achievable over the support of the bidders' valuation distributions

Nearly-matching exponential lower bounds.

How many samples do we need?

Affine maximizer auctions [R79]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

$\cup$ $\cup$

Virtual valuation combinatorial auctions [SL03]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

λ-auctions [J07]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

$\cup$ $\cup$

Mixed bundling auctions with reserve prices [TS12]

$$N = \tilde{O}\left((U/\epsilon)^2 m^3\right)$$

$\cup$

Mixed bundling auctions [J07]

$$N = \tilde{O}\left((U/\epsilon)^2\right)$$

Variables

- N: sample size
- n: # bidders
- m: # items
- U: maximum revenue achievable over the support of the bidders' valuation distributions

Learning theory tool: Rademacher complexity

How many samples do we need?

Affine maximizer auctions [R79]
$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$

$\cup$ $\cup$

Virtual valuation combinatorial auctions [SL03]
$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$

λ-auctions [J07]
$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$

$\cup$ $\cup$

Mixed bundling auctions with reserve prices [TS12]
$N = \tilde{O}\left((U/\epsilon)^2 m^3\right)$

$\cup$

Mixed bundling auctions [J07]
$N = \tilde{O}\left((U/\epsilon)^2\right)$

Variables

- N: sample size
- n: # bidders
- m: # items
- U: maximum revenue achievable over the support of the bidders' valuation distributions

Learning theory tool: Pseudo-dimension

Outline

- Introduction
- Hierarchy of deterministic combinatorial auction classes
- Our contribution: how many samples are needed to learn over the hierarchy of auctions?
- Affine maximizer auctions and Rademacher complexity
- Mixed-bundling auctions and pseudo-dimension
- Summary and future directions

How many samples do we need?

Affine maximizer auctions [R79]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

∪ ∪

Virtual valuation combinatorial auctions [SL03]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

λ-auctions [J07]

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

∪ ∪

Mixed bundling auctions with reserve prices [TS12]

$$N = \widetilde{O}\left((U/\epsilon)^2 m^3\right)$$

∪

Mixed bundling auctions [J07]

$$N = \widetilde{O}\left((U/\epsilon)^2\right)$$

Variables

- N: sample size
- n: # bidders
- m: # items
- U: maximum revenue achievable over the support of the bidders' valuation distributions

Key challenge

Our problem...

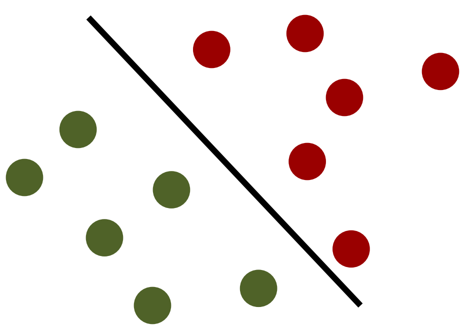
- **Boost outcomes:** $\lambda(o)$; **Weight bidders:** w_i
- **Take bids** v
- **Compute outcome:**

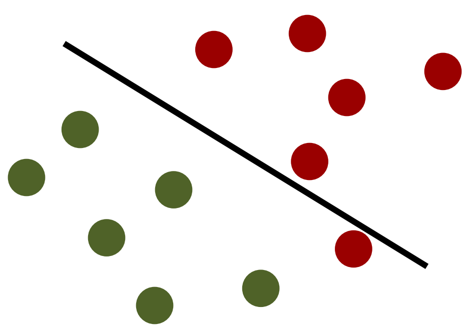
$$o^* = argmax_{\,o}\left\{\sum_{j\in Bidders}^{n} w_j v_j(o) + \lambda(o)\right\}$$

- **Compute Bidder** i**'s payment:**

$$\frac{1}{w_i}\left[\left(\sum_{j\in Bidders-\{i\}} w_j v_j(o^{-i}) + \lambda(o^{-i})\right) - \left(\sum_{j\in Bidders-\{i\}} w_j v_j(o^*) + \lambda(o^*)\right)\right]$$

Whereas typically in machine learning…





Rademacher complexity

- **More expressive function classes need more samples to learn**
- **How to measure expressivity?**
  - **How well do functions from the class fit random noise?**

- **Empirical Rademacher complexity:**

$$(x_1, \dots, x_N) \sim \{-1, 1\}^N, \qquad S = \{v^1, \dots, v^N\}$$

$$R_S(\mathcal{A}) = \mathbb{E}_x\left[sup_{A \in \mathcal{A}} \frac{1}{N} \sum x_i \cdot rev_A(v^i)\right]$$, **where**

- **Rademacher complexity:**

$$R_N(\mathcal{A}) = \mathbb{E}_{S \sim D^N}[R_S(\mathcal{A})]$$

- **With probability at least $1 - \delta$, for all $A \in \mathcal{A}$,**

|empirical revenue – expected revenue| $\leq 2R_N(\mathcal{A}) + U\sqrt{\frac{2\ ln(2/\delta)}{N}}$

***U is the maximum revenue achievable over the support of the bidders' valuation distributions**

Rademacher complexity

- More expressive function classes need more samples to learn
- How to measure expressivity?
  - How well do functions from the class fit random noise?

- **Empirical Rademacher complexity:**

$$(x_1, \dots, x_N) \sim \{-1, 1\}^N, \qquad S = \{v^1, \dots, v^N\}$$

$$R_S(\mathcal{A}) = \mathbb{E}_x\left[sup_{A \in \mathcal{A}} \frac{1}{N} \sum x_i \cdot rev_A(v^i)\right]$$, where

- **Rademacher complexity:**

$$R_N(\mathcal{A}) = \mathbb{E}_{S \sim D^N}[R_S(\mathcal{A})]$$

$\mathcal{A}$ = all binary valued functions	$R_N(\mathcal{A}) = \frac{1}{2}$
$\mathcal{A}$ = one binary valued function	$R_N(\mathcal{A}) = 0$

Rademacher complexity of AMAs

Theorem

Let $\mathcal{A}$ be the class of n-bidder, m-item AMA revenue functions. If

$$N = O\left(\left[Un^m\sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right),$$

then with high probability over a sample $S \sim D^N$,

|empirical revenue – expected revenue| < ϵ for all $rev_A \in \mathcal{A}$.

- **Key idea:** split revenue function into its simpler components
  - Weighted social welfare without any one bidder's participation (n components)
  - Amount of revenue subtracted out to maintain strategy-proof property
- Then use compositional properties of Rademacher complexity and other tricks, for example:

  If $F = \{f \mid f = g + h, g \in G, h \in H\}$, then $R_N(F) \leq R_N(G) + R_N(H)$

Outline

- Introduction
- Hierarchy of deterministic combinatorial auction classes
- Our contribution: how many samples are needed to learn over the hierarchy of auctions?
- Affine maximizer auctions and Rademacher complexity
- Mixed-bundling auctions and pseudo-dimension
- Summary and future directions

How many samples do we need?

Affine maximizer auctions [R79]

$$N = O\left(\left[U n^m \sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

∪ ∪

Virtual valuation combinatorial auctions [SL03]

$$N = O\left(\left[U n^m \sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

λ-auctions [J07]

$$N = O\left(\left[U n^m \sqrt{m}\left(U + n^{m/2}\right)/\epsilon\right]^2\right)$$

∪ ∪

Mixed bundling auctions with reserve prices [TS12]

$$N = \tilde{O}\left((U/\epsilon)^2 m^3\right)$$

∪

Mixed bundling auctions [J07]

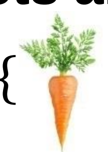
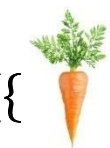
$$N = \tilde{O}\left((U/\epsilon)^2\right)$$

Variables

N: sample size
n: # bidders
m: # items
U: maximum revenue achievable over the support of the bidders' valuation distributions

Mixed bundling auctions (MBAs)

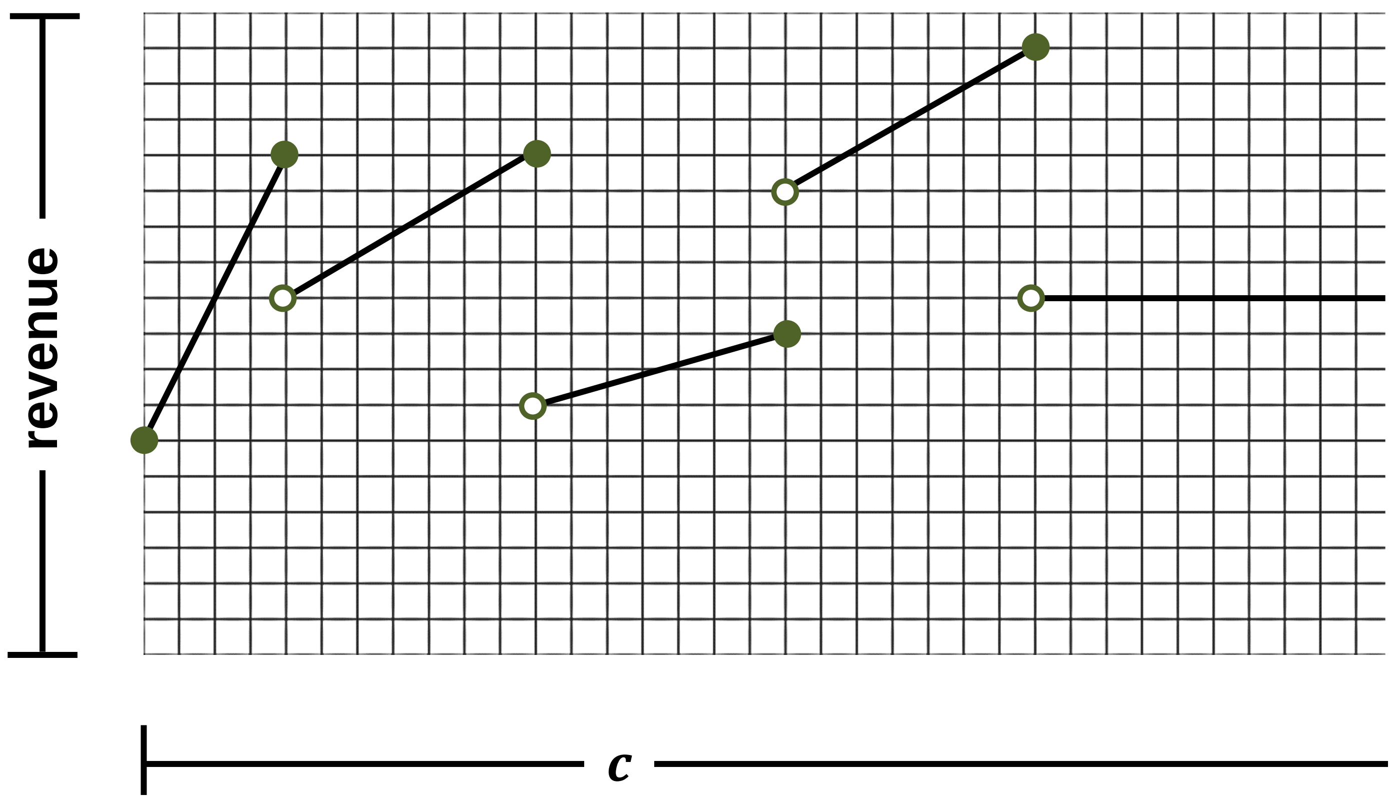
NINA	TUOMAS
 : $1	 : 50¢
 : $0	 : 50¢
  : $1	  : 50¢

- **Class of auctions parameterized by a scalar c**
- **Boost the allocations where one bidder gets all goods by c**
- **value**$(\emptyset, \{$  ,  $\}) = v_{Nina}(\emptyset) + v_{Tuomas}(\{$  ,  $\})$ **= 50¢ + 99¢**
- **value**$(\{$  ,  $\}, \emptyset) = v_{Nina}(\{$  ,  $\}) + v_{Tuomas}(\emptyset)$ **= 50¢ + 99¢**
- **How large must the sample S be in order to ensure that for all MBAs, |empirical revenue – expected revenue| < ϵ?**

Structural properties of MBA revenue functions

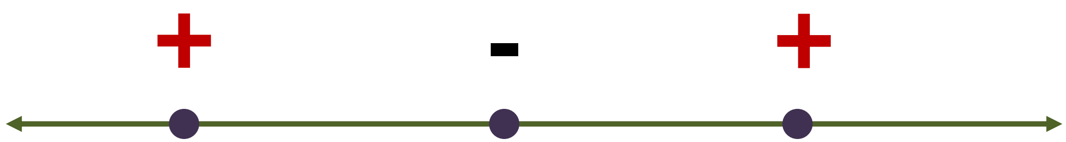
Lemma

Fix $v \in S$. Then $rev_v(c)$ is piecewise linear with at most $n + 1$ discontinuities.



VC-dimension

- Complexity measure for binary-valued functions only
- Example: $\boldsymbol{F}$ = {single interval on the real line}
- No set of size 3 can be labeled in all 2^3 ways by $\boldsymbol{F}$



- Class of functions $\boldsymbol{F}$ shatters set $\boldsymbol{S} = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ if
  for all b $\in \{0, 1\}^N$, there exists $\boldsymbol{f} \in \boldsymbol{F}$ such that $\boldsymbol{f}(\boldsymbol{x}_i) = \boldsymbol{b}_i$
- VC-dimension of $\boldsymbol{F}$ is the
  cardinality of the largest set $\boldsymbol{S}$ that can be shattered by $\boldsymbol{F}$

How can we extend VC-dim to real-valued functions?

Pseudo-dimension

x_1	$f(x_1)$	$\leq$	$r^{(1)}$	0
x_2	$f(x_2)$	$\leq$	$r^{(2)}$	0
x_3	$f(x_3)$	$>$	$r^{(3)}$	1
x_4	$f(x_4)$	$\leq$	$r^{(4)}$	0
x_5	$f(x_5)$	$>$	$r^{(5)}$	1
x_6	$f(x_6)$	$\leq$	$r^{(6)}$	0
x_7	$f(x_7)$	$>$	$r^{(7)}$	1
x_8	$f(x_8)$	$>$	$r^{(8)}$	1
x_9	$f(x_9)$	$\leq$	$r^{(9)}$	0

- **Sample $S = \{x_1, \dots, x_N\}$**
- **Class of functions F into $[-U, U]$**
- $r = \left(r^{(1)}, \dots, r^{(N)}\right) \in \mathbb{R}^N$ **witnesses the shattering of S by F if for all $T \subseteq S$, there exists $f_T \in F$ such that $f_T(x_i) \leq r^{(i)}$ iff $x_i \in T$**
- **Pseudo-dimension of F is the cardinality of the largest sample S that can be shattered by F**

P-dim(F) = VC-dim($\{(x, r) \mapsto \mathbf{1}_{f(x)-r>0} \mid f \in F\}$)

How many samples do we need?

- **Set of auction revenue functions $\mathcal{A}$ with range in $[0, U]$, distribution D over valuations v.**
- **For every $\epsilon > 0$, $\delta \in (0, 1)$, if**

$$N = O\left(\left(\frac{U}{\epsilon}\right)^2 \left(\text{P-dim}(\mathcal{A}) * \ln\frac{U}{\epsilon} + \ln\frac{1}{\delta}\right)\right),$$

then with probability at least $1 - \delta$ over a sample $S \sim D^N$,

|empirical revenue – expected revenue| $< \epsilon$

for every $rev_A \in \mathcal{A}$.

Pseudo-dimension allows us to derive strong sample complexity bounds.

How many samples do we need?

Affine maximizer auctions [R79]

$$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$$

$\cup$ $\cup$

Virtual valuation combinatorial auctions [SL03]

$$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$$

λ-auctions [J07]

$$N = O\left(\left[Un^m\sqrt{m}(U + n^{m/2})/\epsilon\right]^2\right)$$

$\cup$ $\cup$

Mixed bundling auctions with reserve prices [TS12]

$$N = \tilde{O}\left((U/\epsilon)^2 m^3\right)$$

$\cup$

Mixed bundling auctions [J07]

$$N = \tilde{O}\left((U/\epsilon)^2\right)$$

Variables

- N: sample size
- n: # bidders
- m: # items
- U: maximum revenue achievable over the support of the bidders' valuation distributions

2-bidder MBA pseudo-dimension

Theorem

Let $\mathcal{A} = \{rev_c\}_{c \geq 0}$ be the class of n-bidder, m-item mixed bundling auction revenue functions. Then P-dim($\mathcal{A}$) $= O(\log n)$.

2-bidder MBA pseudo-dimension

Theorem

Let $\mathcal{A} = \{rev_c\}_{c \geq 0}$ be the class of 2-bidder, m-item mixed bundling auction revenue functions. Then P-dim($\mathcal{A}$) $= 2$.

Proof sketch.

- **Fact:** there exists a set of 2 samples that is shattered by $\mathcal{A}$.

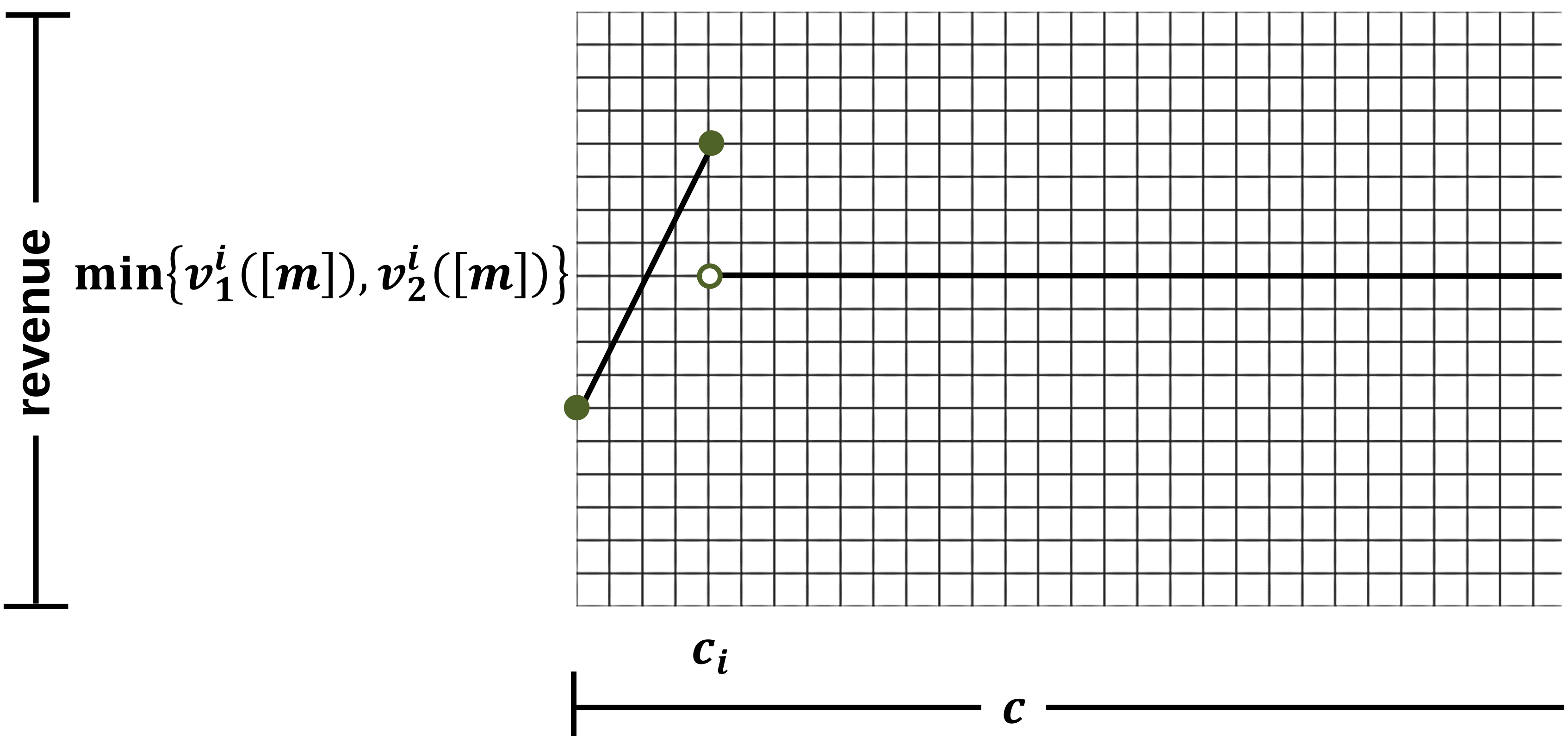
We need to show that no set of 3 samples can be shattered by $\mathcal{A}$.

- Suppose, for a contradiction, that $S = \{v^1, v^2, v^3\}$ is shatterable.
- Recall $v^1 = (v_1^1, v_2^1)$
- This means:
  - There exists $r = (r^1, r^2, r^3) \in \mathbb{R}^3$ and $2^{|S|} = 8$ MBA parameters $\mathrm{C} = \{c_1, \ldots, c_8\}$ such that $\{rev_{c_1}, \ldots, rev_{c_8}\}$ induce all 8 binary labelings on S with respect to r.

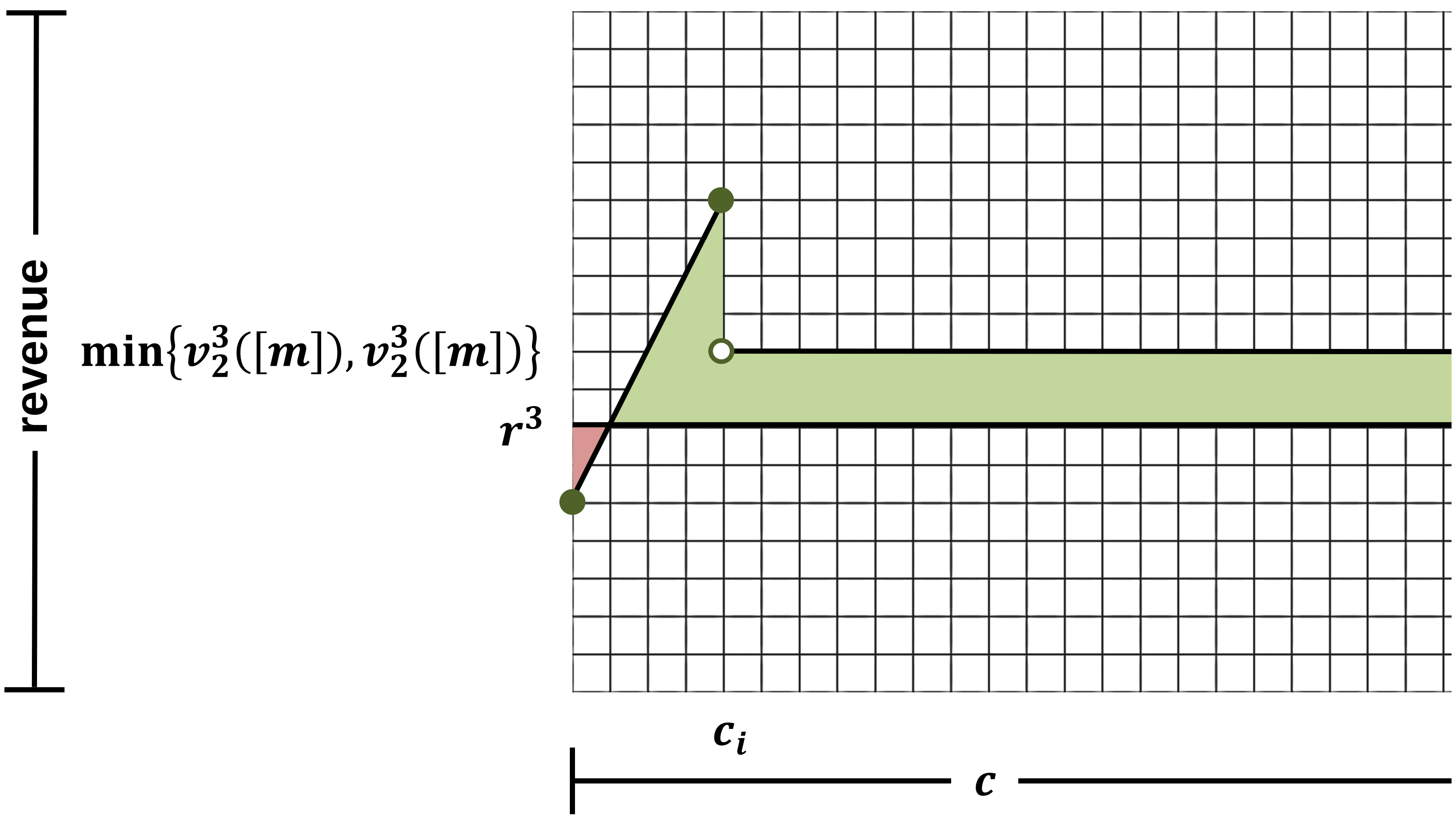
2-bidder MBA pseudo-dimension

Lemma

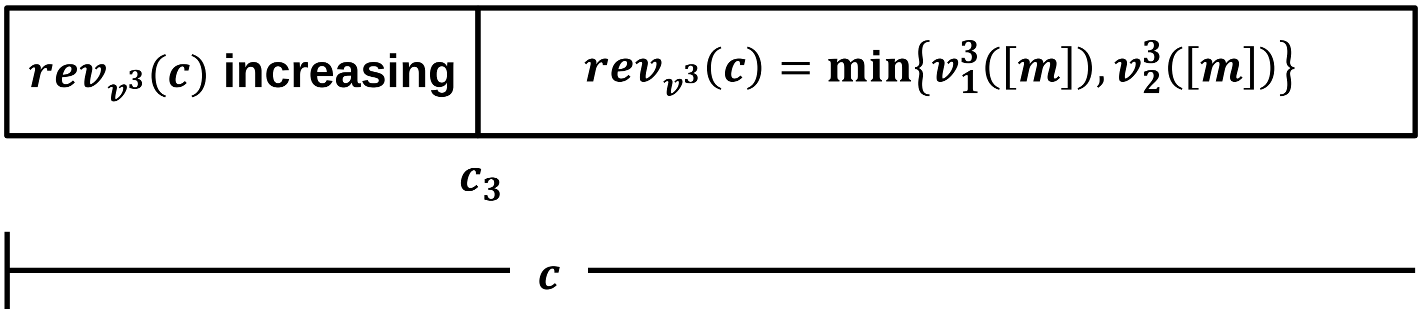
Fix $v^i \in S$. Then $rev_{v^i}(c)$ is piecewise linear with one discontinuity, with a slope of 2 followed by a constant function with value $\min\{v_1^i([m]), v_2^i([m])\}$.



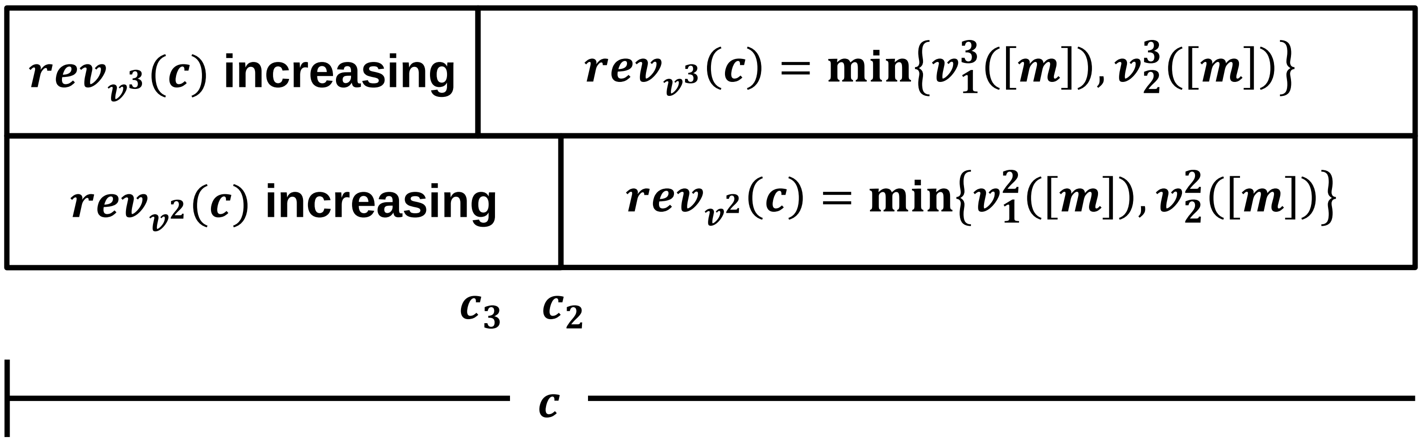
Case 1: $r^3 < min\{v_1^3([m]), v_2^3([m])\}$



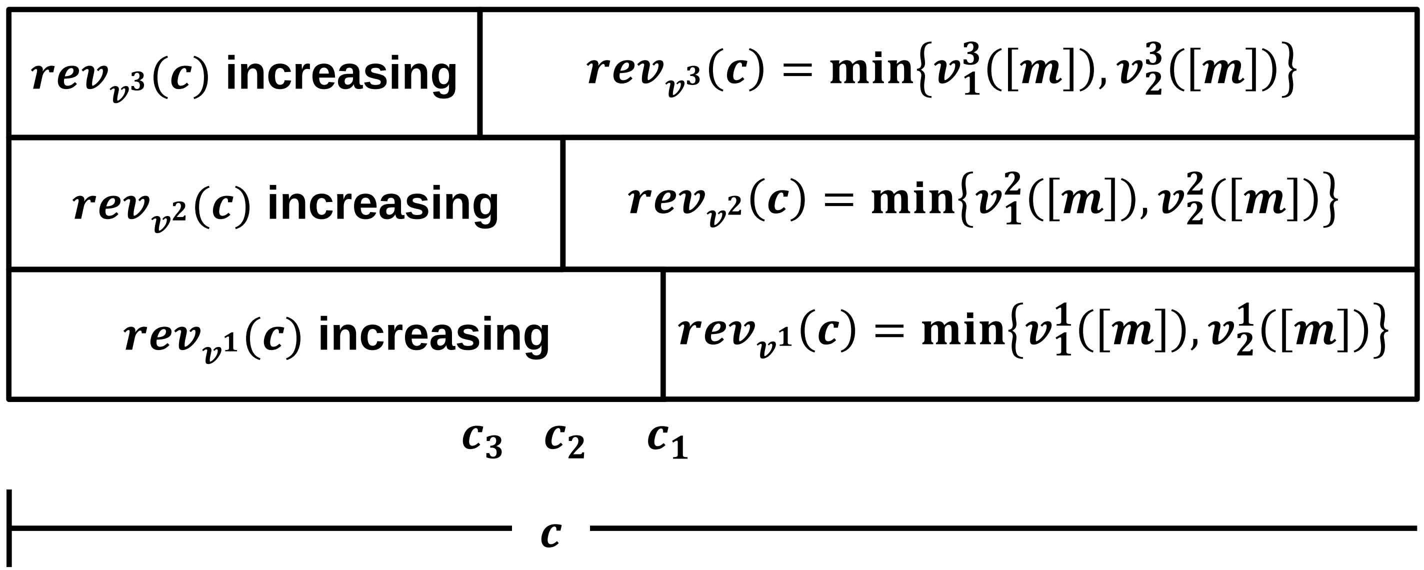
Case 1: $r^3 < min\{v_1^3([m]), v_2^3([m])\}$



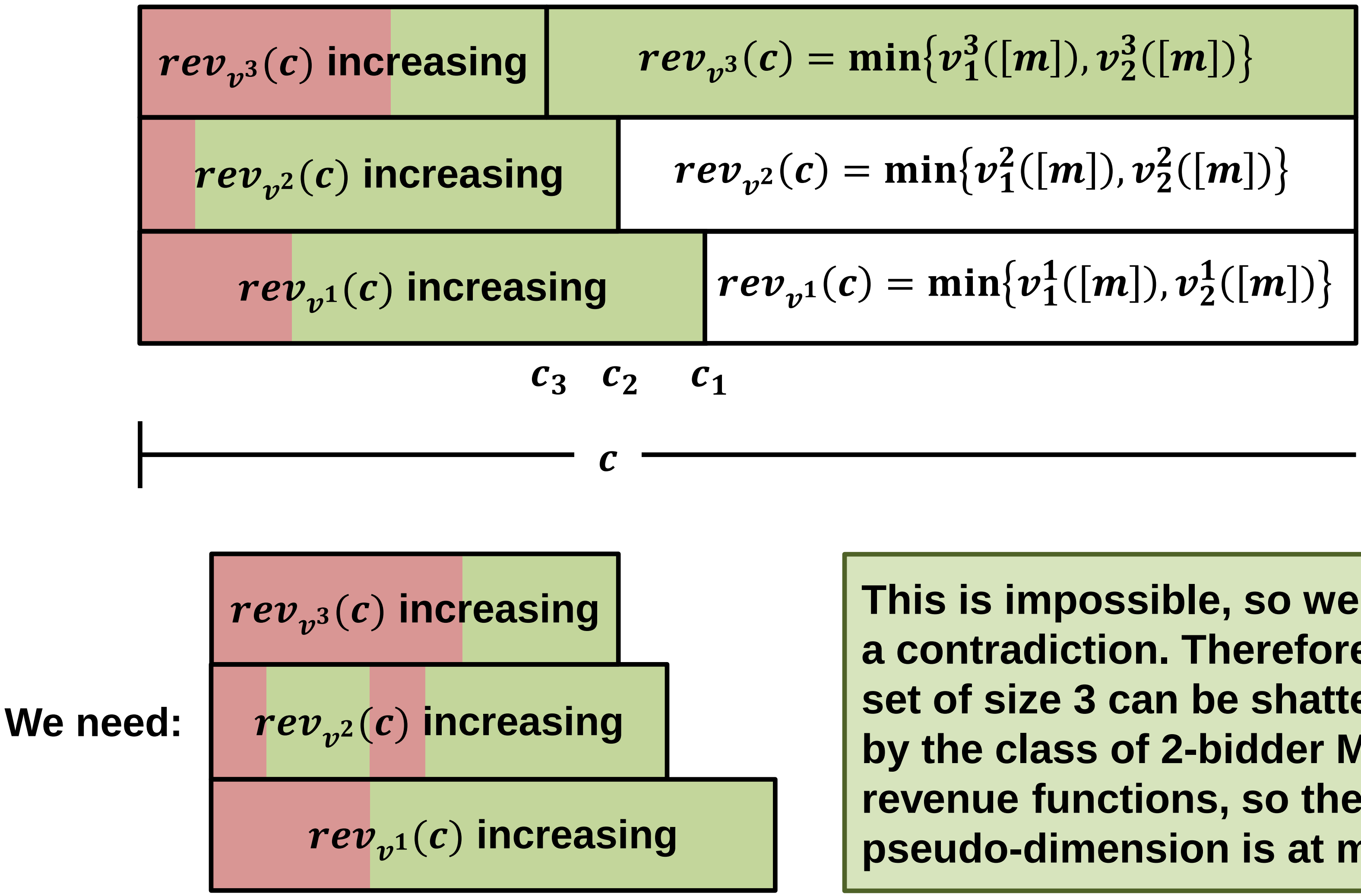
Case 1: $r^3 < min\{v_1^3([m]), v_2^3([m])\}$



Case 1: $r^3 < min\{v_1^3([m]), v_2^3([m])\}$



Case 1: $r^3 < min\{v_1^3([m]), v_2^3([m])\}$



This is impossible, so we reach a contradiction. Therefore, no set of size 3 can be shattered by the class of 2-bidder MBA revenue functions, so the pseudo-dimension is at most 2.

Summary

- Analyzed the sample complexity of learning over a hierarchy of deterministic combinatorial auctions
- Uncovered structural properties of these auctions' revenue functions along the way
  - Of independent interest beyond sample complexity results