Discrete optimization crash course

Content draws on material from Optimization Methods in Management Science from MIT Sloan

An important property of algorithms used in practice is

broad applicability

Example: **Integer programming solvers**

Most popular tool for solving combinatorial (& nonconvex) problems

Routing

Manufacturing

Scheduling

Planning

Finance

…but they can have **unsatisfactory** default performance

Slow runtime, poor solutions, …

Integer programming (IP)

IP solvers (CPLEX, Gurobi) have a **ton** of parameters

- CPLEX has **170-page** manual describing **172** parameters
- Tuning by hand is notoriously **slow**, **tedious**, and **error-prone**

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What's the best **configuration** for the application at hand?

Best configuration for **routing** problems
likely not suited for **scheduling**

Plan

This class: Overview of how these solvers work

Future classes: How to use ML to optimize these solvers

Outline

1. **Linear programming**
2. Integer programming
3. SAT solving
4. Next steps

Linear programming

Linear programming (LP) is a central topic in optimization

Provides a powerful tool for modeling many applications

Tons of attention over past two decades due to:

- **Applicability:** Many real-world applications can be modeled via LPs
- **Solvability:** Efficient techniques for solving large-scale problems

Basic components of an LP

Each optimization problem consists of 3 elements:

- **Decision variables**: describe our choices that are under our control;
- **Objective function**: Criterion that we wish to minimize (e.g., cost)
  or maximize (e.g., profit)
- **Constraints**: Limitations restricting our choices for decision variables

"Linear programming" refers to an optimization problem where:

- The **objective function** is linear
- Each **constraint** is a linear inequality or equality

An introductory example

A company makes two products (say, P and Q)

- Uses two machines (say, A and B)

Each unit of P that is produced requires:

- 50 minutes processing time on machine A, and
- 30 minutes processing time on machine B

Each unit of Q that is produced requires:

- 24 minutes processing time on machine A, and
- 33 minutes processing time on machine B

An introductory example

- Machine A is going to be available for 40 hours
- Machine B is available for 35 hours
- Profit per unit of P is $28
- Profit per unit of Q is $30
- **Goal:** determine production quantity of P and Q such that:
    1. Total profit is maximized
    2. Available resources aren't exceeded
- **Task:** formulate this problem as an LP

Step 1: Defining the decision variables

Decision variables: Describe choices under our control

Goal: determine production quantity of P and Q such that …

So there are 2 decision variables:

x: the number of units of P

y: the number of units of Q

Step 2: Choosing an objective function

Usually seek a criterion to compare alternative solutions

This yields the objective function

Want to maximize the total profit

- Profit per each unit of product P is $28
- Profit per each unit of Q is $30

Total profit is $28x + 30y$ if we produce x units of P & y units of Q

Leads to the following objective function:

$$\max \underbrace{28x + 30y}_{\textbf{Linear}}$$

Step 3: Identifying the constraints

Often are limitations that restrict our decisions

Resource, physical, strategic, economical

We describe these limitations using **mathematical constraints**

Step 3: Identifying the constraints

- Each unit of P requires 50 minutes on machine A
- Each unit of Q requires 24 minutes on machine A
- If we produce x units of P and y units of Q:
  - Machine A needs to be used for $50x + 24y$
- Machine A is available for 40 hours = 2400 minutes
  - This imposes the following constraint: $50x + 24y \leq 2400$
- Similarly, amount of time machine B is available means that:

$$30x + 33y \leq 2100$$

Step 3: Identifying the constraints

In most problems, decision variables must be nonnegative

So need to include the following two constraints as well:

$x \geq 0$ and $y \geq 0$

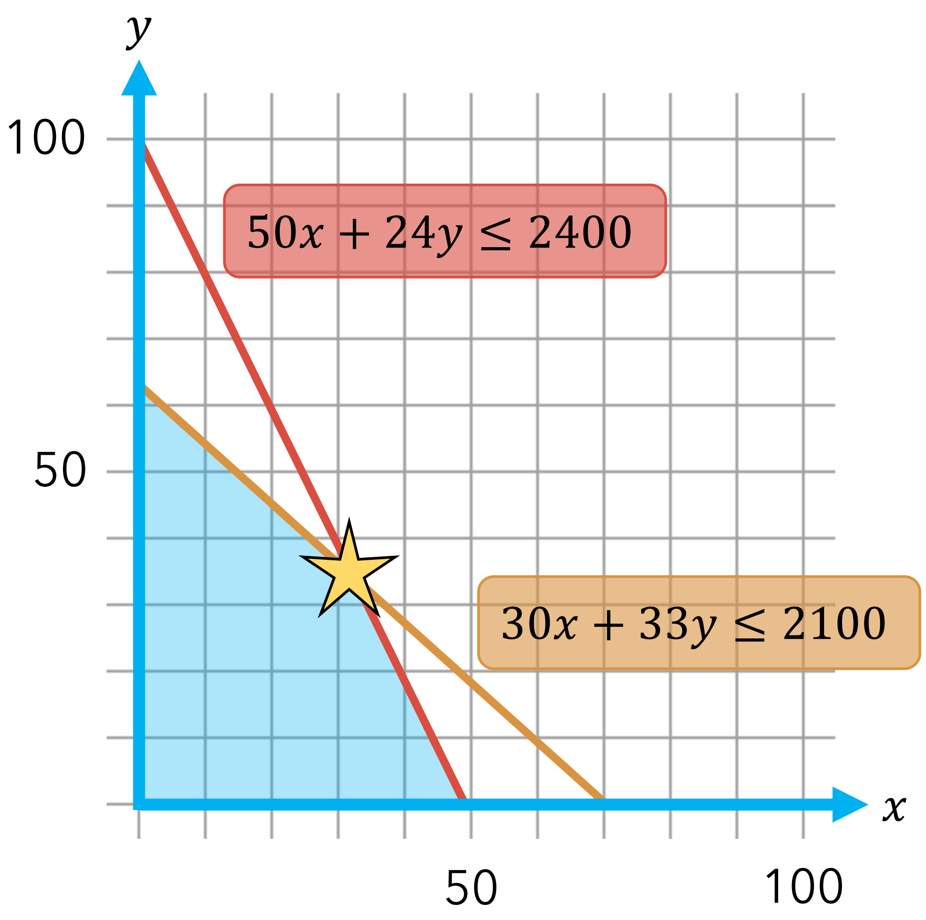
In the end, the constraints we're subject to (s.t.) are :

$50x + 24y \leq 2400$, (machine A time)
$30x + 33y \leq 2100$, (machine B time)
$x \geq 0$,
$y \geq 0$

LP for the example

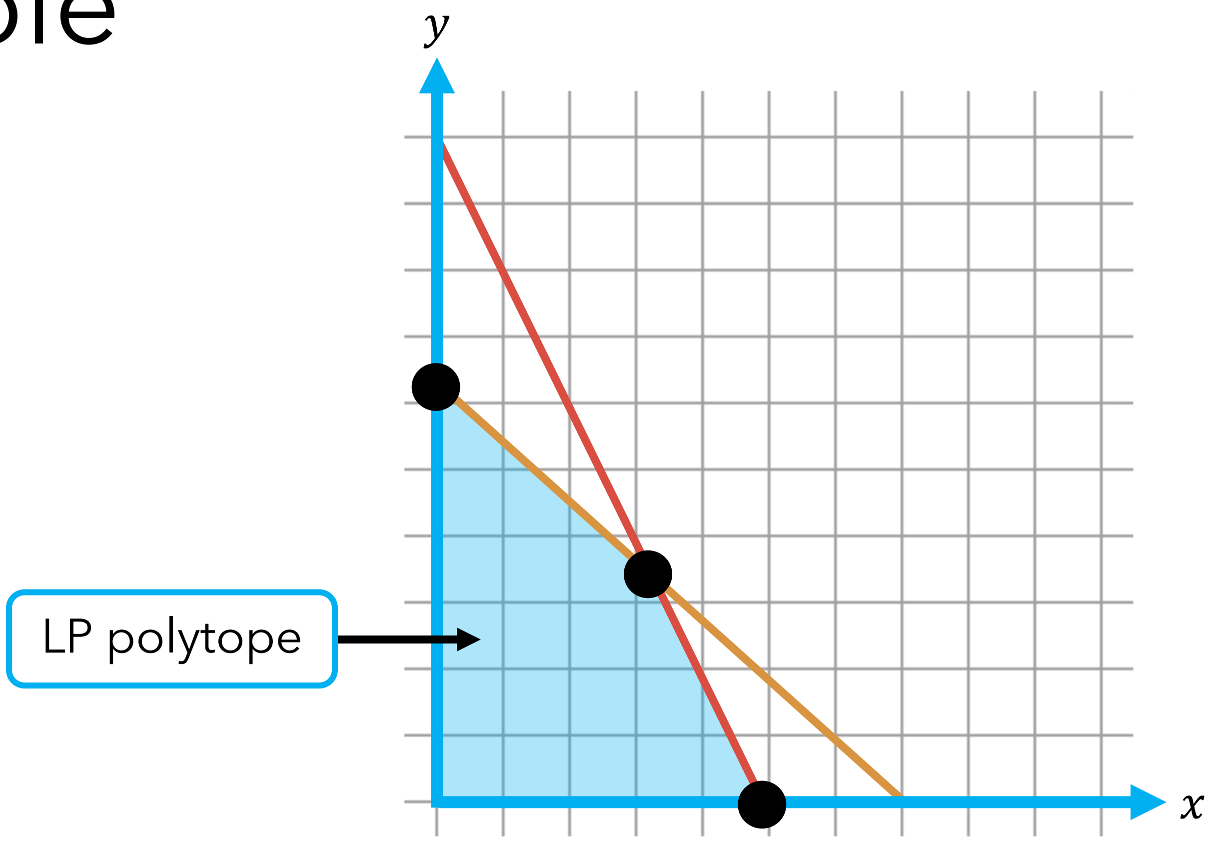
Here is the LP:

$$\begin{array}{ll} \text{maximize} & 28x + 30y \\ \text{subject to} & 50x + 24y \leq 2400 \\ & 30x + 33y \leq 2100 \\ & x \geq 0 \text{ and } y \geq 0 \end{array}$$



Optimal solution: $x = 30.97, y = 35.48$

LP for the example



Fact: Optimal solution of an LP is always at a vertex

LP algorithms

Simplex algorithm: Practical algorithm for solving LPs

- May run in exponential time in the worst case
- Provable runs in polynomial time on “realistic” LPs
- Used by commercial solvers like CPLEX, Gurobi, …

Ellipsoid method: Impractical but provably runs in poly-time

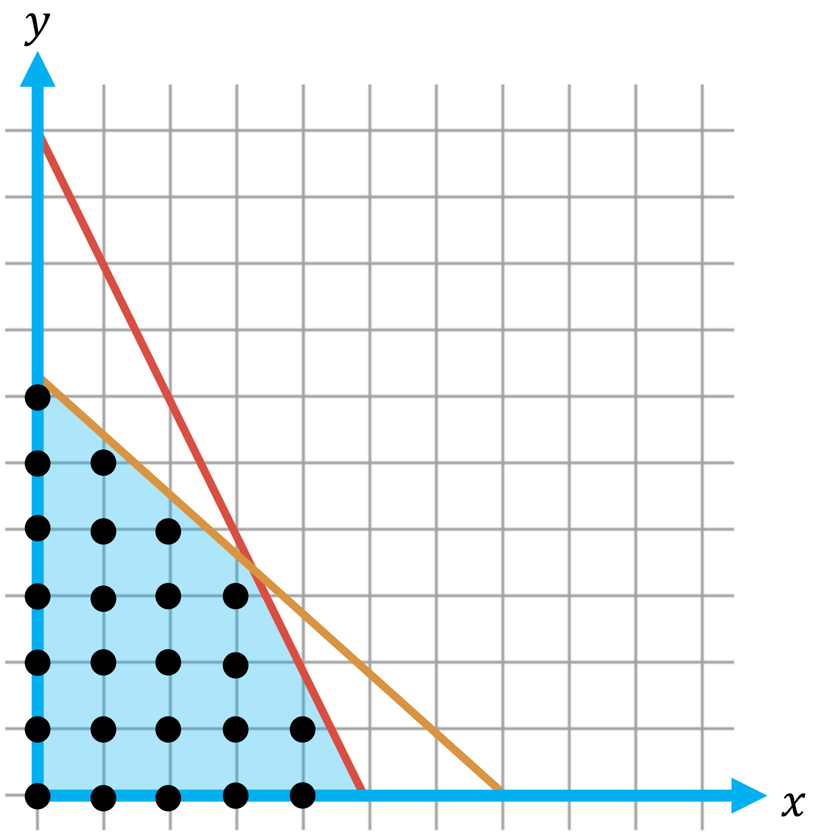
Outline

1. Linear programming
2. **Integer programming**
3. SAT solving
4. Next steps

Integer programming (IP)

What if the decision variables must be integral?

$$\begin{array}{ll} \text{maximize} & 28x + 30y \\ \text{subject to} & 50x + 24y \leq 2400 \\ & 30x + 33y \leq 2100 \\ & x \geq 0 \text{ and } y \geq 0 \\ & \boldsymbol{x, y \in \mathbb{Z}} \end{array}$$

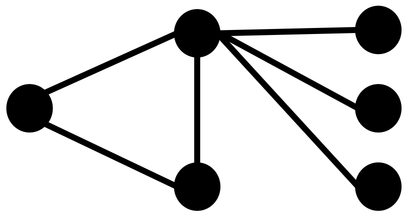


Integer programming is NP-complete

Example: vertex cover

Vertex cover of a graph:

Set of vertices that includes ≥ 1 endpoint of every edge



Example: vertex cover

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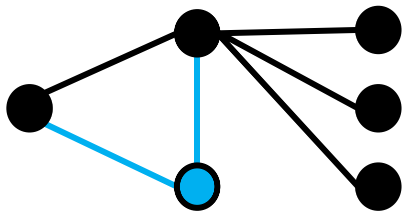
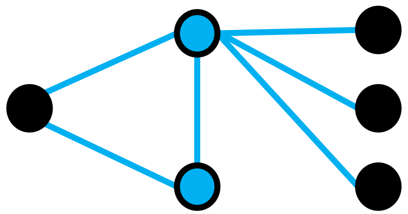


Example: vertex cover

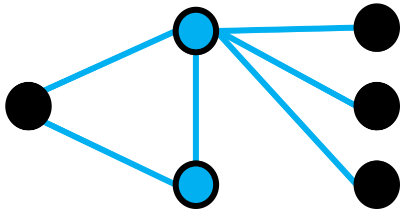
Vertex cover of a graph:

Set of vertices that includes ≥ 1 endpoint of every edge



Goal: Find a vertex cover of minimal size

Vertex cover IP



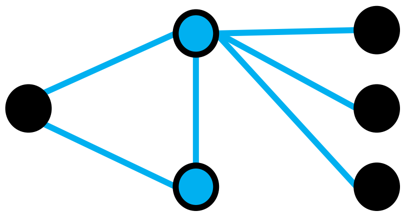
Input: Graph $G = (V, E)$ with vertex set V, edge set E

1. Decision variables: For each vertex $v \in V$,
$$y_v = \begin{cases} 1 & \text{if } v \text{ in vertex cover} \\ 0 & \text{otherwise} \end{cases}$$
2. Objective function: minimize $\sum_{v \in V} y_v$
3. Constraints:

   For every edge $(u, v) \in E$, need $y_u = 1$ and/or $y_v = 1$

   In other words: $y_u + y_v \geq 1$

Vertex cover IP

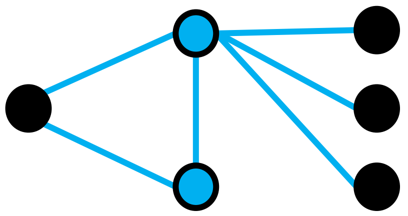


Input: Graph $G = (V, E)$ with vertex set V, edge set E

minimize $\sum_{v \in V} y_v$
subject to $y_u + y_v \geq 1$ for all $(u, v) \in E$
$\underbrace{y_v \in \{0,1\}}$ for all $v \in V$

***Binary* integer program**

LP relaxations



Input: Graph $G = (V, E)$ with vertex set V, edge set E

minimize $\sum_{v \in V} y_v$

subject to $y_u + y_v \geq 1$ for all $(u, v) \in E$

$0 \leq y_v \leq 1$

$\cancel{y_v \in \{0,1\}}$

If you remove the integrality constraints,
you obtain the ***LP relaxation*** of the IP

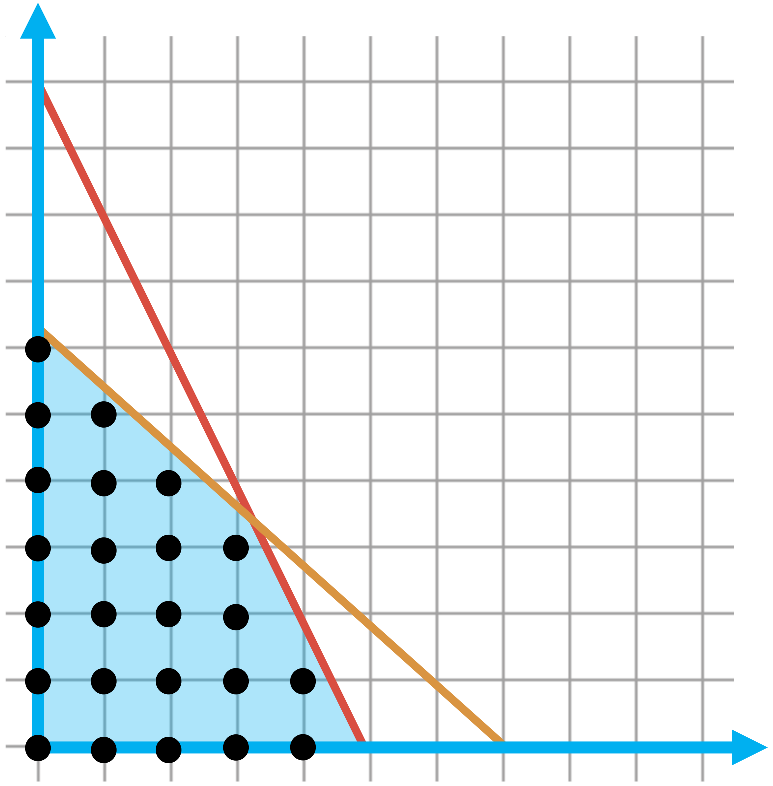
LP relaxations

Integer program

$$\begin{array}{ll} \max & \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \in \mathbb{Z}^n \end{array}$$

LP relaxation

$$\begin{array}{ll} \max & \boldsymbol{c} \cdot \boldsymbol{x} \\ \text{s.t.} & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \cancel{\boldsymbol{x} \in \mathbb{Z}^n} \end{array}$$



$\boldsymbol{x}_{IP}^*$ = optimal solution to IP
$\boldsymbol{x}_{LP}^*$ = optimal solution to LP relaxation
Fact: $\boldsymbol{c} \cdot \boldsymbol{x}_{IP}^* \leq \boldsymbol{c} \cdot \boldsymbol{x}_{LP}^*$

LP relaxations

Integer program

min $\boldsymbol{c} \cdot \boldsymbol{x}$

s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$

$\boldsymbol{x} \in \mathbb{Z}^n$

LP relaxation

min $\boldsymbol{c} \cdot \boldsymbol{x}$

s.t. $A\boldsymbol{x} \leq \boldsymbol{b}$

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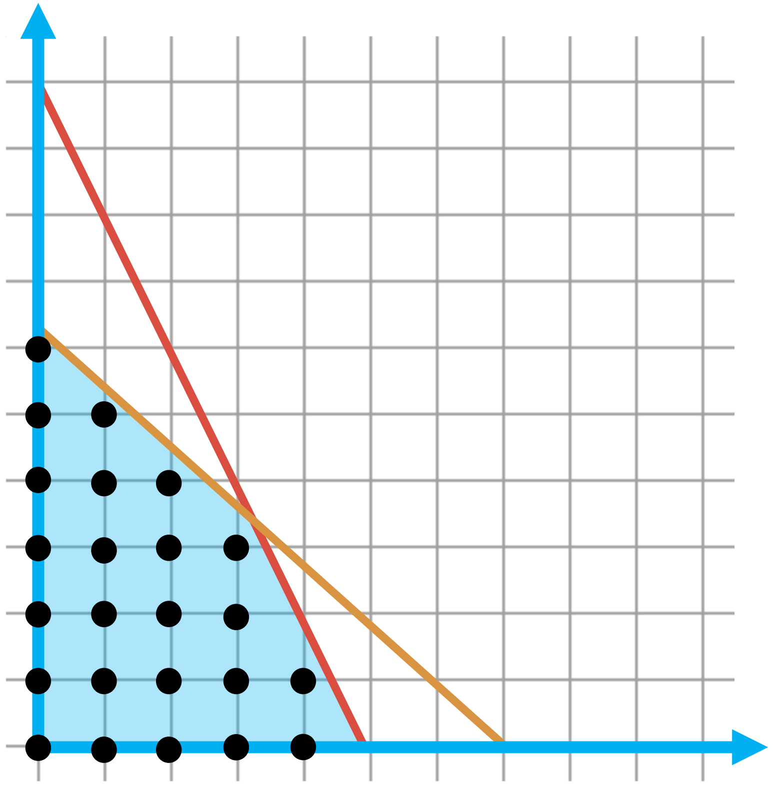


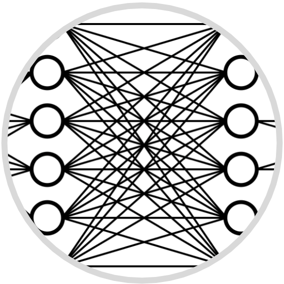
$\boldsymbol{x}^*_{IP}$ = optimal solution to IP

$\boldsymbol{x}^*_{LP}$ = optimal solution to LP relaxation

Fact: $\boldsymbol{c} \cdot \boldsymbol{x}^*_{IP} \geq \boldsymbol{c} \cdot \boldsymbol{x}^*_{LP}$

Integer programming solvers

Most popular tool for solving combinatorial problems



Robust ML

Routing

Manufacturing

Scheduling

Planning

Branch-and-bound

maximize $15x_1 + 12x_2 + 4x_3 + 2x_4$

subject to $8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$

$x_1, x_2, x_3, x_4 \in \{0,1\}$

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
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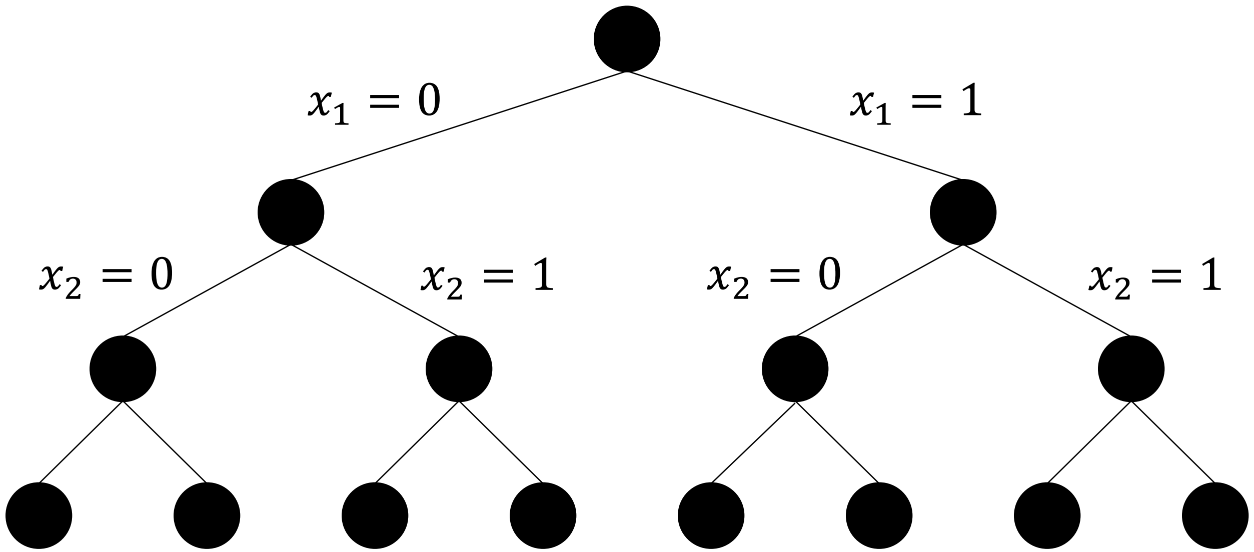
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

- *Enumeration tree*: enumerates all possible solutions of an IP
- At each node, *branch* on an integer variable
  - On each branch, integer variable is restricted to take certain values



Branch-and-bound

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- *Enumeration tree*: enumerates all possible solutions of an IP
- If we can enumerate all solutions with the tree, why not compute objective for each solution and pick the best one?

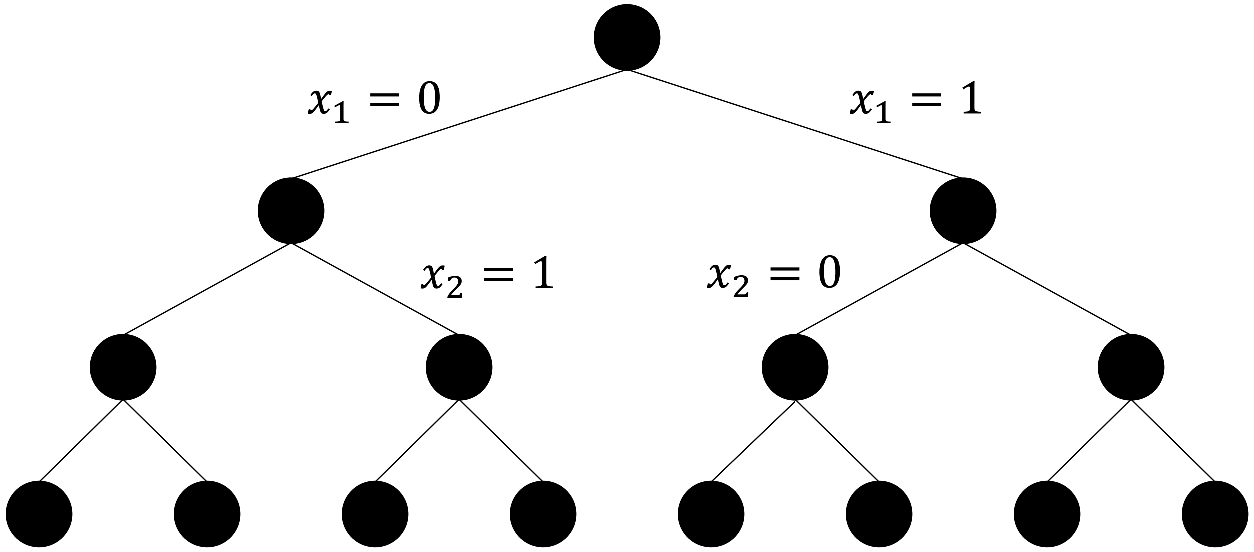
- Would work, but # possible solutions explodes exponentially

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

Key idea of branch-and-bound (B&B):

- Using **LP relaxations**, bound the optimal integer solutions in **subtrees** of the enumeration tree



Branch-and-bound

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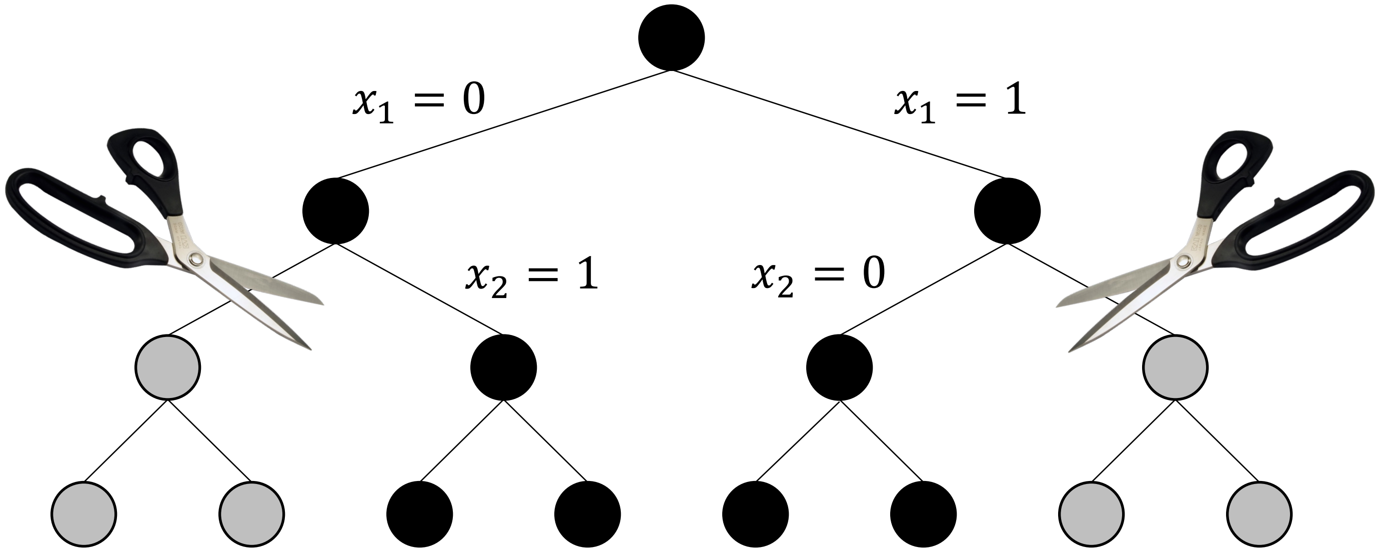
Key idea of branch-and-bound (B&B):

- Using **LP relaxations**, bound the optimal integer solutions in **subtrees** of the enumeration tree
- Allows us to eliminate a lot of the enumeration tree



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	$x_1, x_2, x_3, x_4 \in \{0,1\}$

To start, we assume we have a feasible solution $\boldsymbol{x}^*$

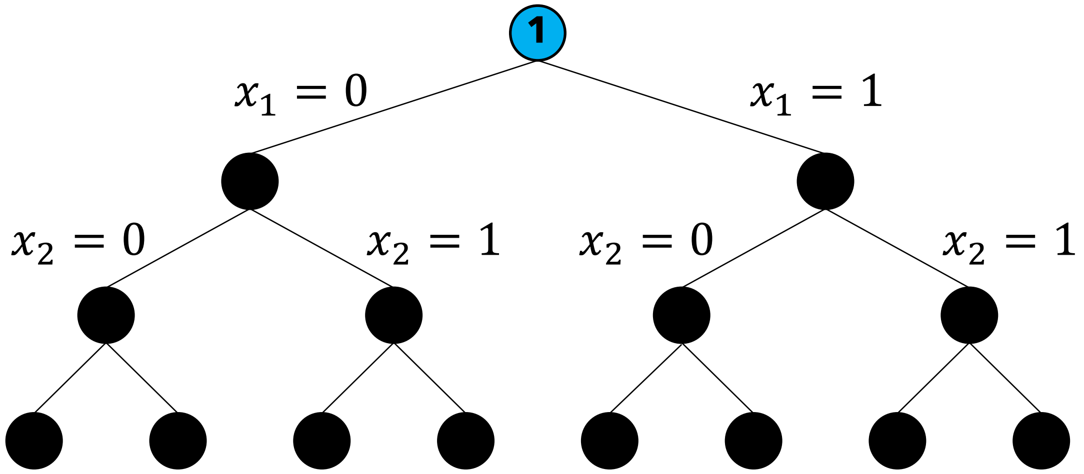
- E.g., $\boldsymbol{x}^* = (0,0,0,0)$

At each iteration of B&B:

- $\boldsymbol{x}^*$ is the *incumbent* solution
- Its objective value z^* is the *incumbent objective*

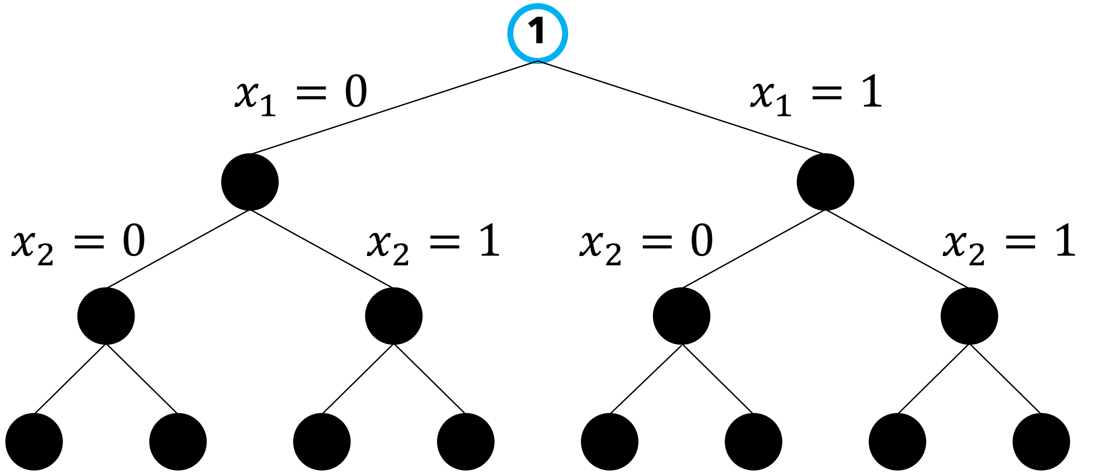
Here, incumbent means “best so far”

Branch-and-bound



1. Mark the root node as active
2. While there remain active nodes:

Branch-and-bound

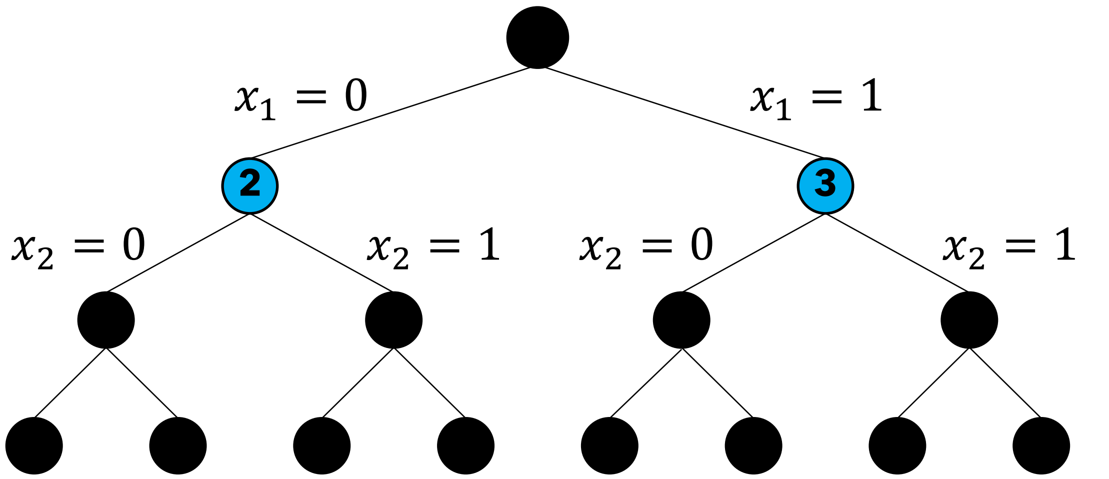


1. Mark the root node as active
2. While there remain active nodes:
    i. Select an active node j and mark it as inactive
    ii. $\boldsymbol{x}(j)$ = optimal solution of LP relaxation of Problem(j)
    iii. $z(j)$ = objective value of $\boldsymbol{x}(j)$
    iv. Case 1: If $z^* < z(j)$ and $\boldsymbol{x}(j)$ isn't feasible for IP then

        Mark the direct descendants of node j as active

        Possible to find a better incumbent solution among j's descendants

Branch-and-bound



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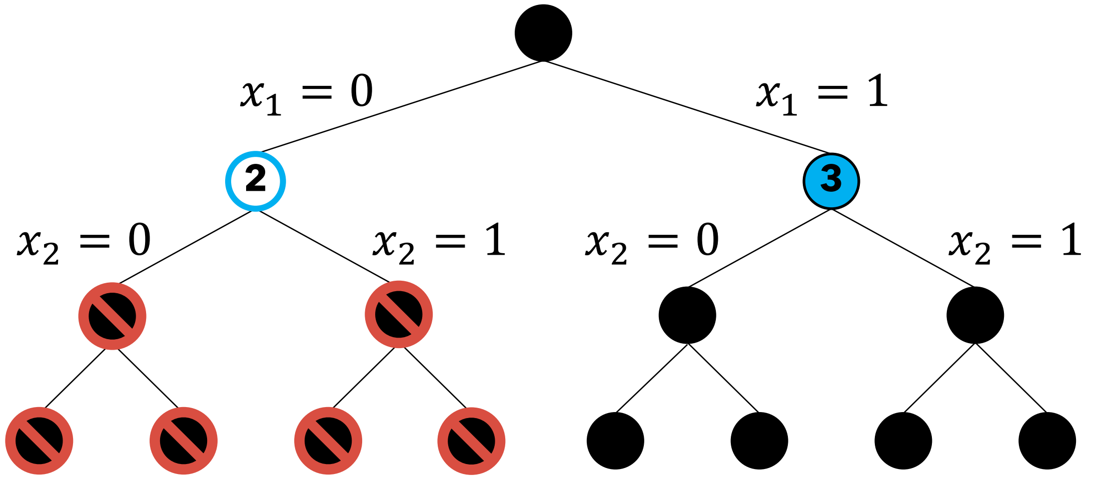

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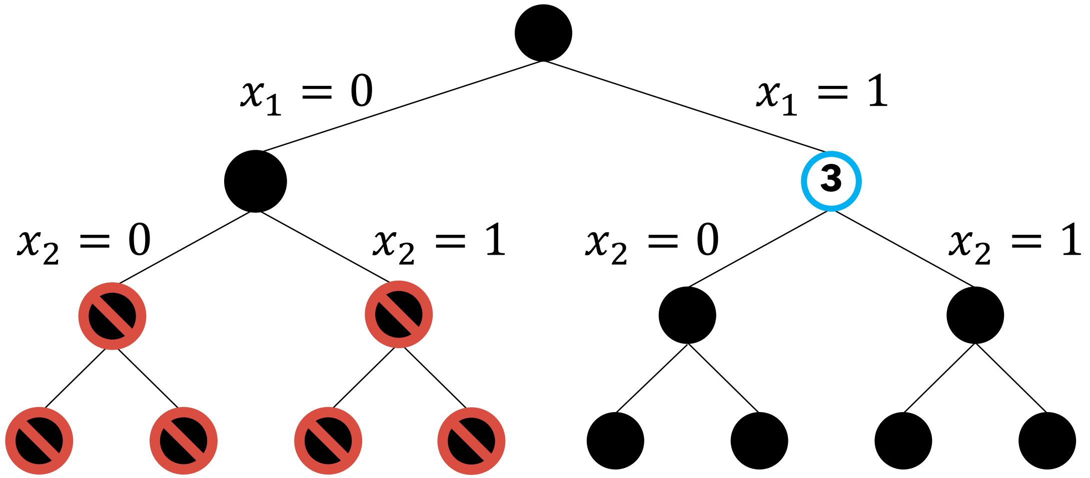
   v. Case 2: If $z^* < z(j)$ and $\boldsymbol{x}(j)$ is feasible for IP then
   Replace the incumbent by $\boldsymbol{x}(j)$ and prune node j
   Could be the optimal solution!

Branch-and-bound



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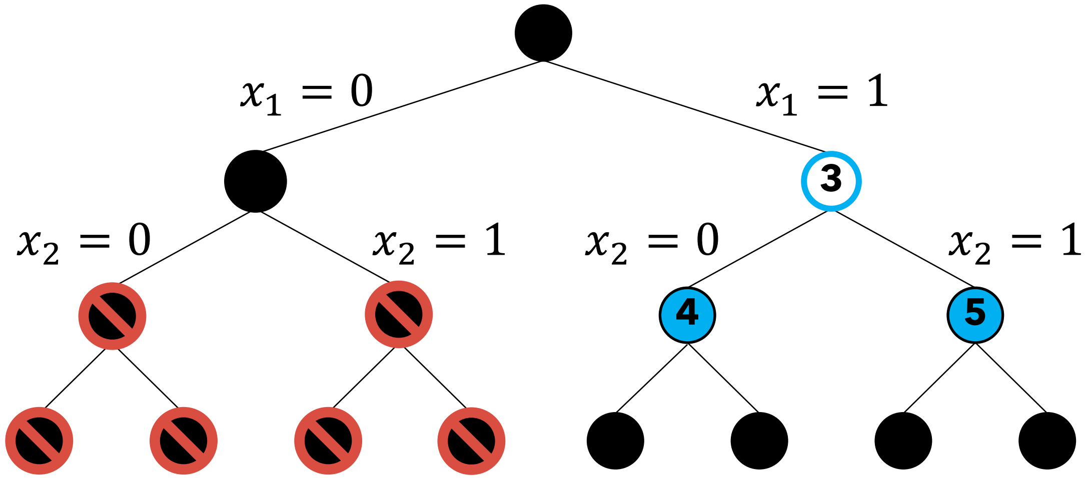

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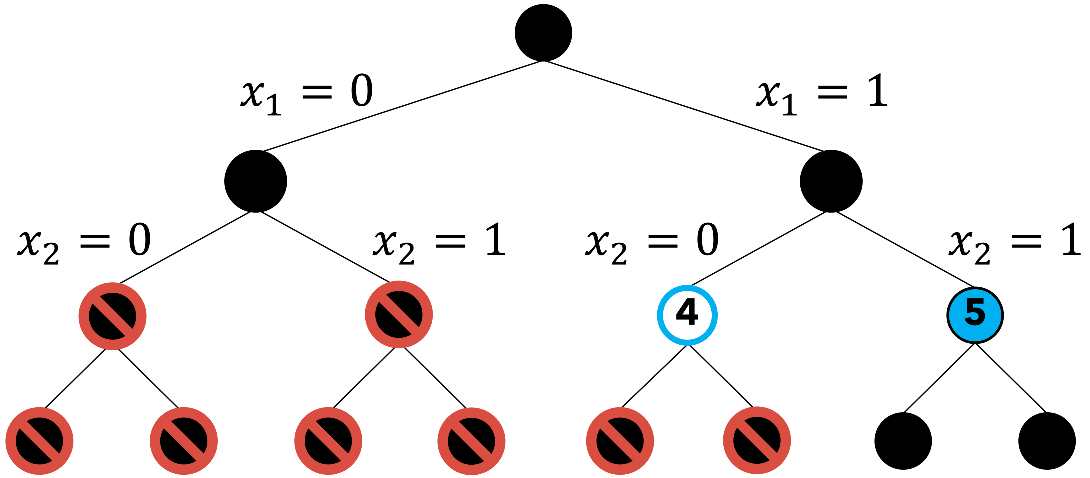

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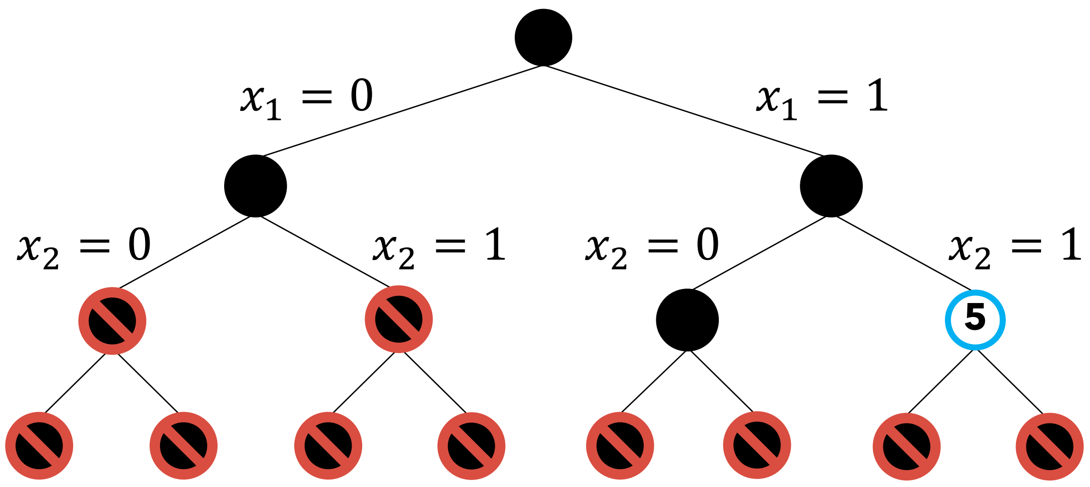
Branch-and-bound



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Branch-and-bound

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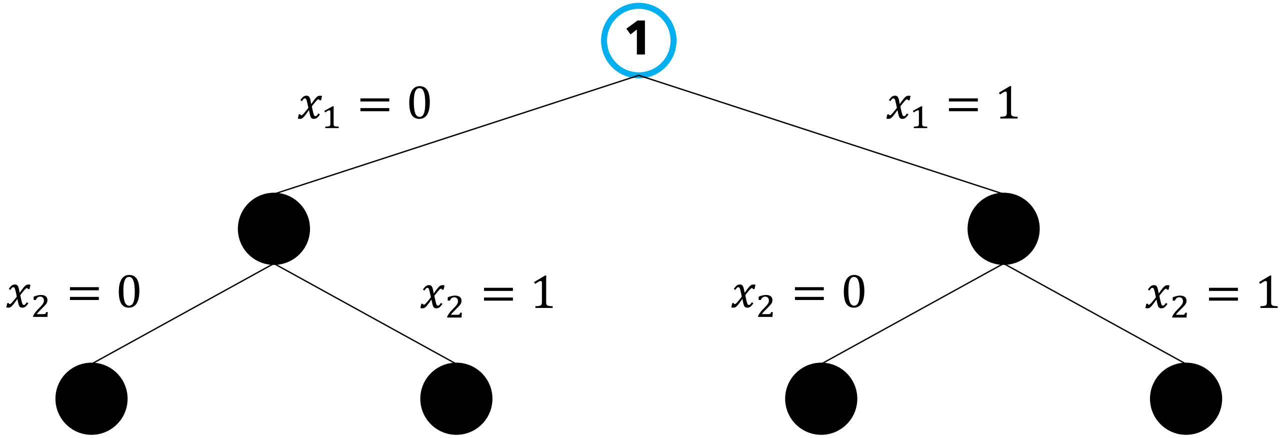
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$



Incumbent: $\boldsymbol{x}^* = (0,0,0,0)$
$z^* = 0$

Problem(1):

max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in [0,1]$

$\boldsymbol{x}(1) = \left(\frac{5}{8}, 1,0,0\right)$
$z(1) = 21.38$

Case 1: If $z^* < z(j)$ and $\boldsymbol{x}(j)$ isn't feasible for IP then
Mark the direct descendants of node j as active

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
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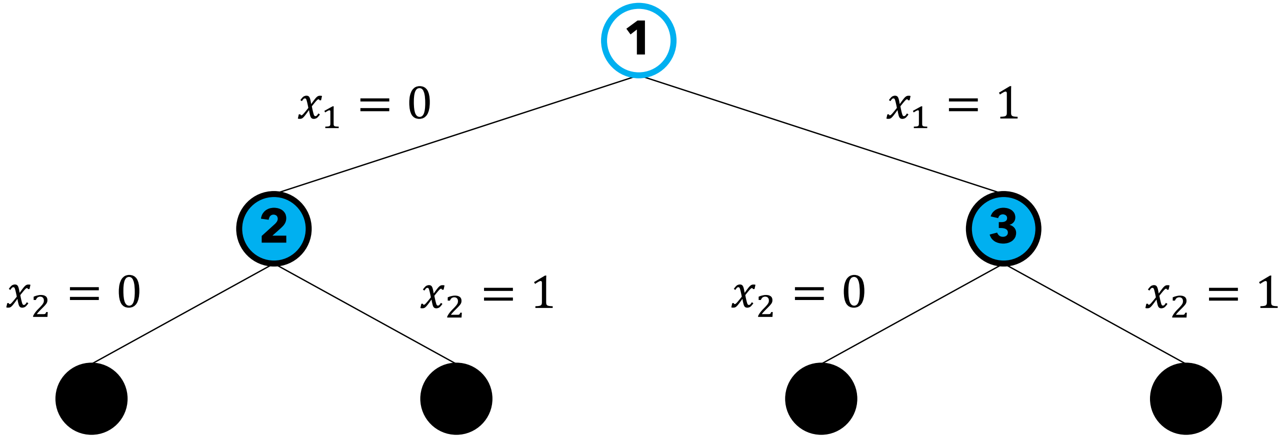

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	$x_1, x_2, x_3, x_4 \in [0,1]$

$\boldsymbol{x}(1) = \left(\frac{5}{8}, 1,0,0\right)$

$z(1) = 21.38$

Case 1: If $z^* < z(j)$ and $\boldsymbol{x}(j)$ isn't feasible for IP then
Mark the direct descendants of node j as active

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

Incumbent: $\boldsymbol{x}^* = (0,0,0,0)$
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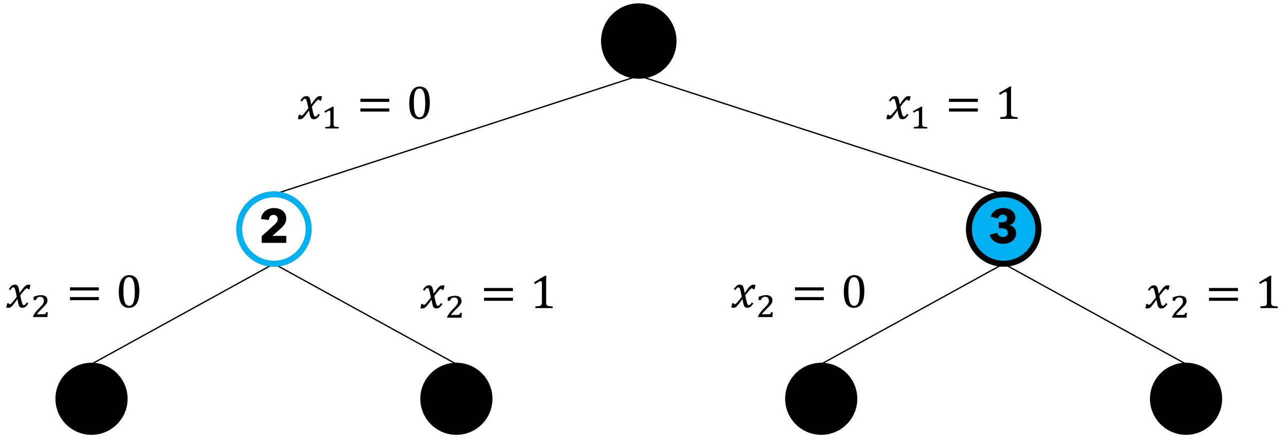


Problem(2):

max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1 = 0$
	$x_2, x_3, x_4 \in [0,1]$

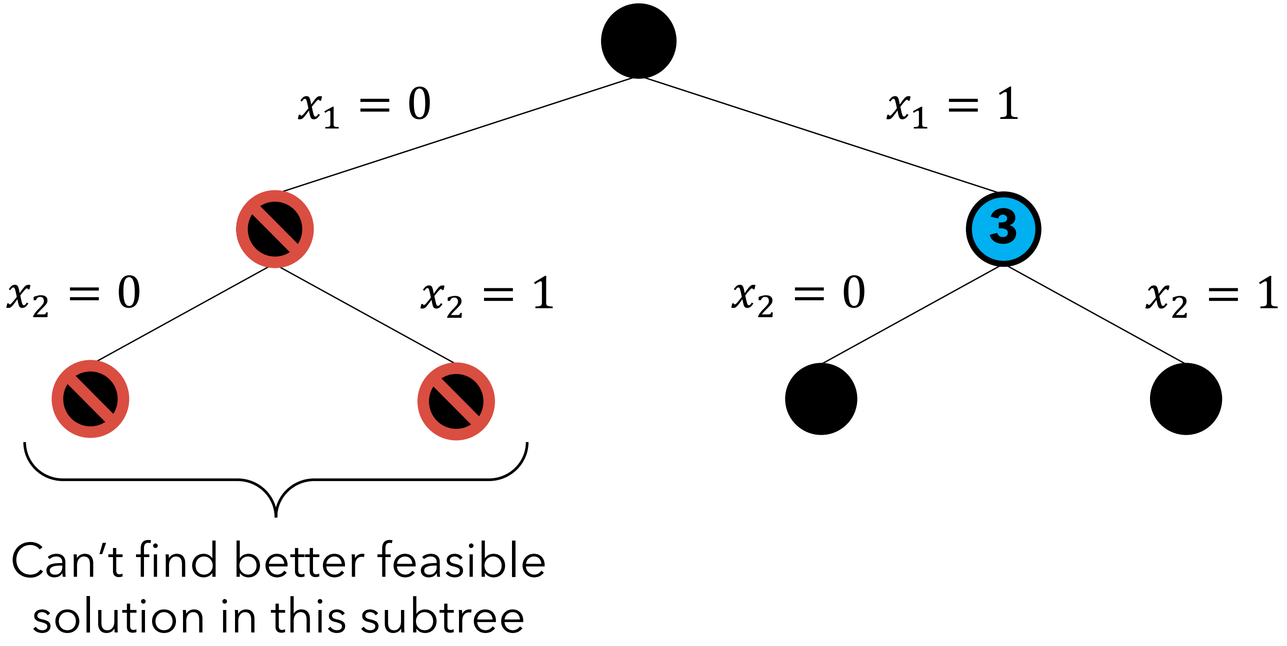
$\boldsymbol{x}(2) = (0,1,1,1)$
$z(2) = 18$

Case 2: If $z^* < z(j)$ and $\boldsymbol{x}(j)$ is feasible for IP then
Replace the incumbent by $\boldsymbol{x}(j)$ and prune node j

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

Incumbent: $\boldsymbol{x}^* = (0,1,1,1)$
$z^* = 18$



Problem(2):

max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1 = 0$
	$x_2, x_3, x_4 \in [0,1]$

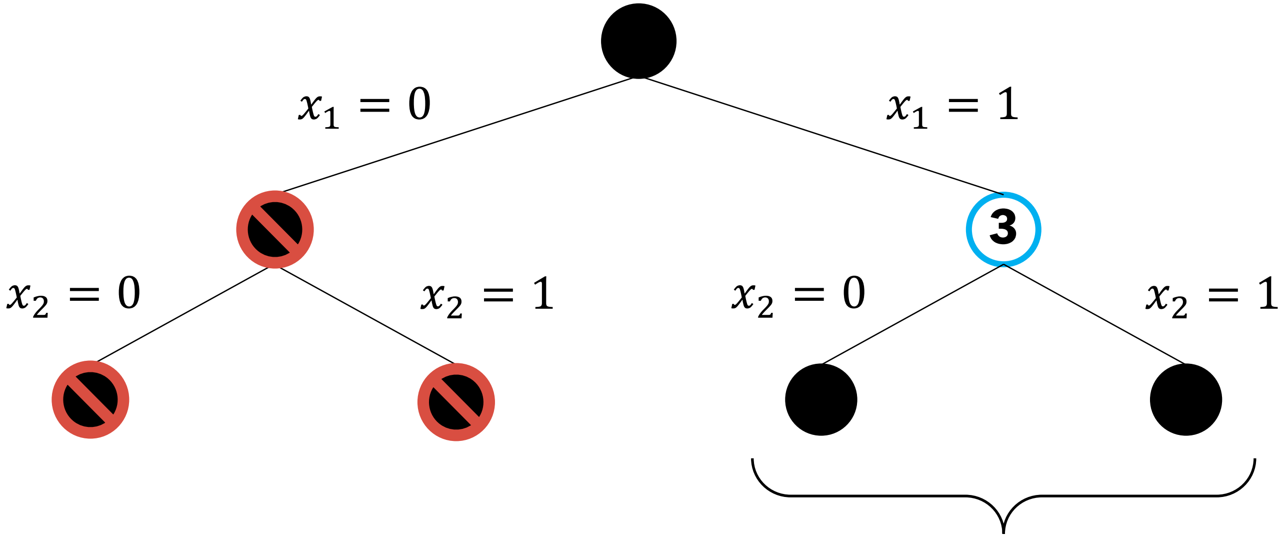
$\boldsymbol{x}(2) = (0,1,1,1)$
$z(2) = 18$

Case 2: If $z^* < z(j)$ and $\boldsymbol{x}(j)$ is feasible for IP then Replace the incumbent by $\boldsymbol{x}(j)$ and prune node j

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

Incumbent: $\boldsymbol{x}^* = (0,1,1,1)$
$z^* = 18$



Could possibly find better feasible solution in this subtree

Problem(3):

max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1 = 1$
	$x_2, x_3, x_4 \in [0,1]$

$\boldsymbol{x}(3) = \left(\frac{5}{8}, 1,0,0\right)$
$z(3) = 21.38$

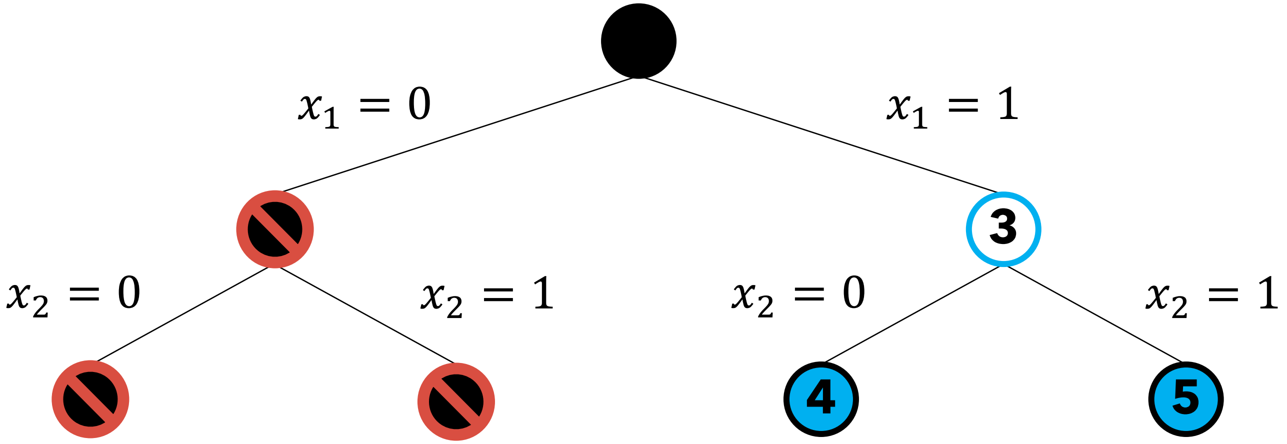
Case 1: If $z^* < z(j)$ and $\boldsymbol{x}(j)$ isn't feasible for IP then
Mark the direct descendants of node j as active

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

Incumbent: $\boldsymbol{x}^* = (0,1,1,1)$

$z^* = 18$



Problem(3):

max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1 = 1$
	$x_2, x_3, x_4 \in [0,1]$

$\boldsymbol{x}(3) = \left(\frac{5}{8}, 1,0,0\right)$

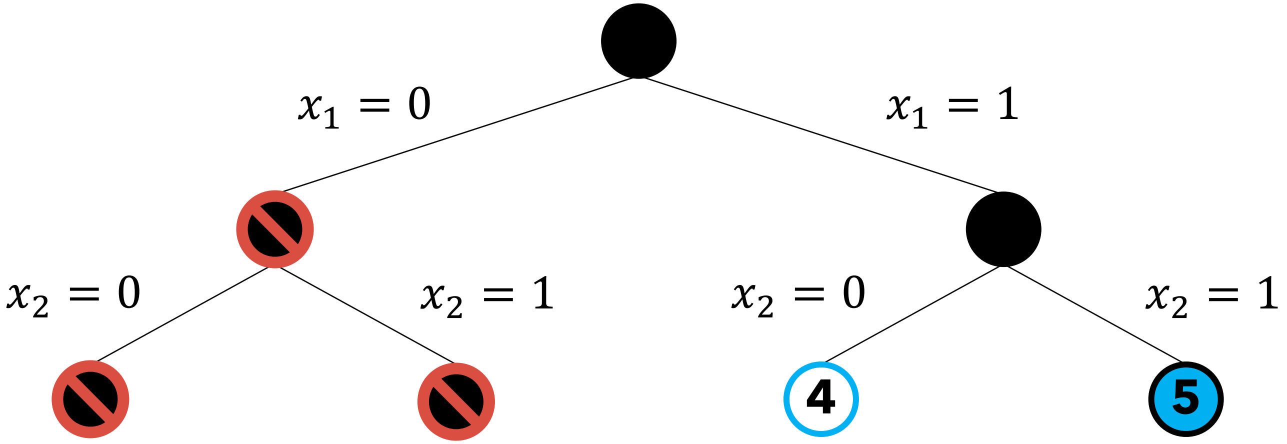
$z(3) = 21.38$

Case 1: If $z^* < z(j)$ and $\boldsymbol{x}(j)$ isn't feasible for IP then
Mark the direct descendants of node j as active

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

Incumbent: $\boldsymbol{x}^* = (0,1,1,1)$
$z^* = 18$



Problem(4):

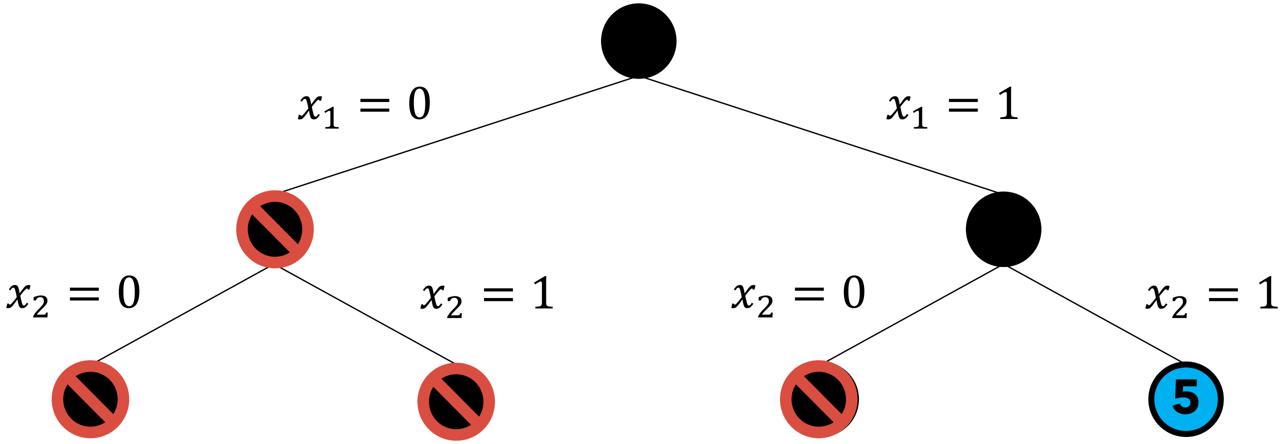
max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1 = 1$
	$x_2 = 0$
	$x_3, x_4 \in [0,1]$

$\boldsymbol{x}(4) = \left(1,0,\frac{2}{3},0\right)$
$z(4) = 17.66$

Case 3: If LP is infeasible or $z^* \geq z(j)$ then prune node j

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$



Incumbent: $\boldsymbol{x}^* = (0,1,1,1)$
$z^* = 18$

Problem(4):

max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1 = 1$
	$x_2 = 0$
	$x_3, x_4 \in [0,1]$

$\boldsymbol{x}(4) = \left(1,0,\frac{2}{3},0\right)$
$z(4) = 17.66$

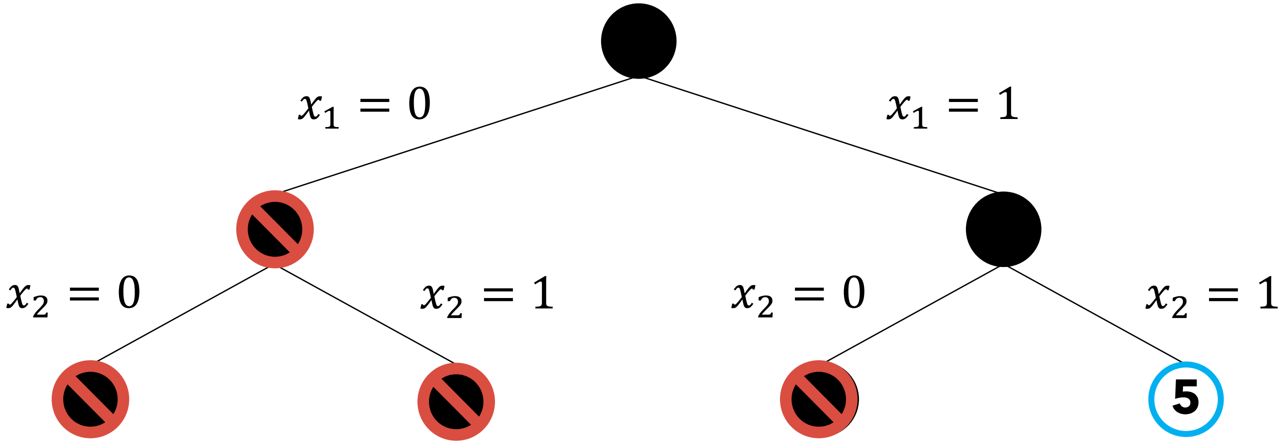
Case 3: If LP is infeasible or $z^* \geq z(j)$ then prune node j

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

Incumbent: $\boldsymbol{x}^* = (0,1,1,1)$

$z^* = 18$



Problem(5):

max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1 = 1$
	$x_2 = 1$
	$x_3, x_4 \in [0,1]$

$\boldsymbol{x}(5) =$ infeasible

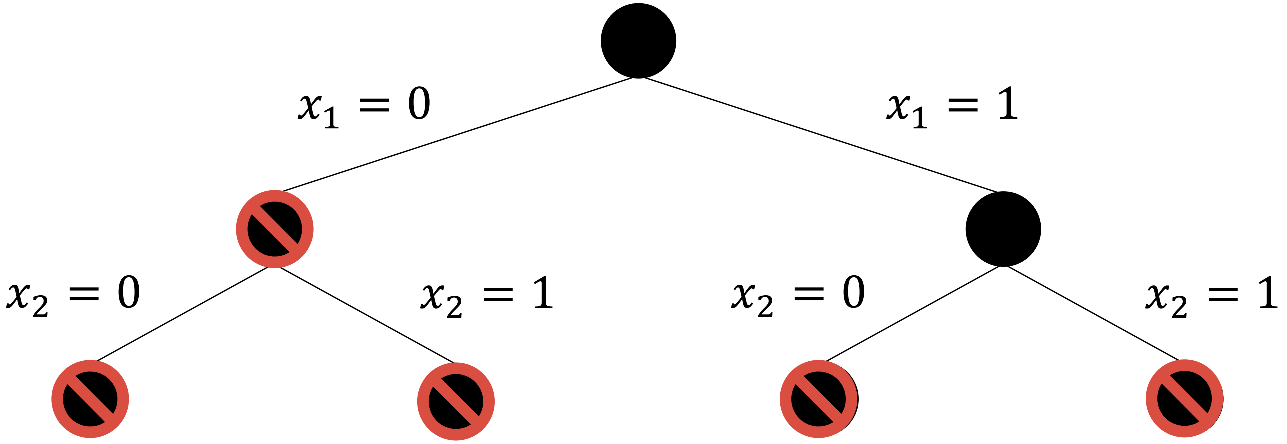
$z(5) =$ infeasible

Case 3: If LP is infeasible or $z^* \geq z(j)$ then prune node j

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$

Incumbent: $\boldsymbol{x}^* = (0,1,1,1)$
$z^* = 18$



Problem(5):

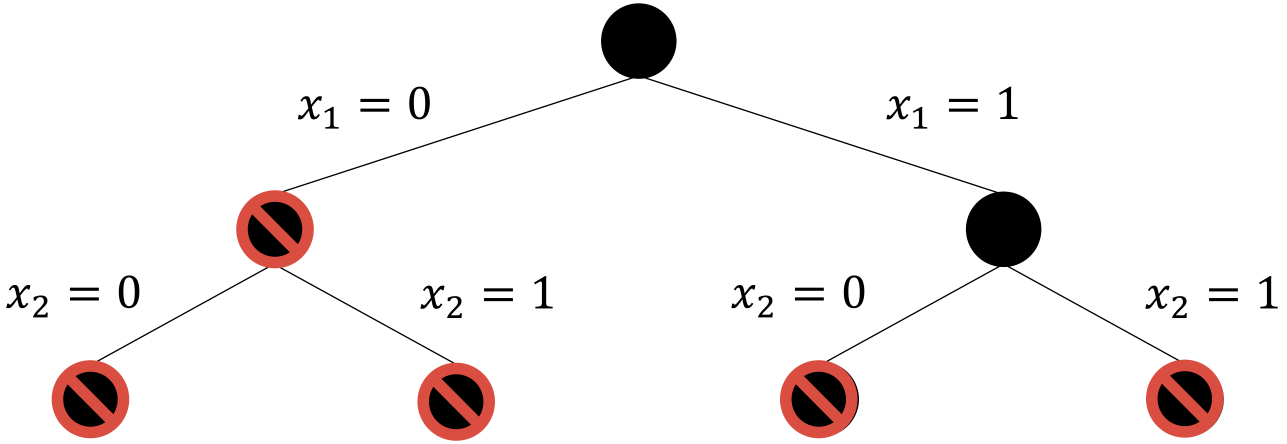
max	$15x_1 + 12x_2 + 4x_3 + 2x_4$
s.t	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1 = 1$
	$x_2 = 1$
	$x_3, x_4 \in [0,1]$

$\boldsymbol{x}(5) =$ infeasible
$z(5) =$ infeasible

Case 3: If LP is infeasible or $z^* \geq z(j)$ then prune node j

Branch-and-bound

maximize	$15x_1 + 12x_2 + 4x_3 + 2x_4$
subject to	$8x_1 + 5x_2 + 3x_3 + 2x_4 \leq 10$
	$x_1, x_2, x_3, x_4 \in \{0,1\}$



Incumbent: $\boldsymbol{x}^* = (0,1,1,1)$
$z^* = 18$

Optimal solution

Major challenge of using B&B

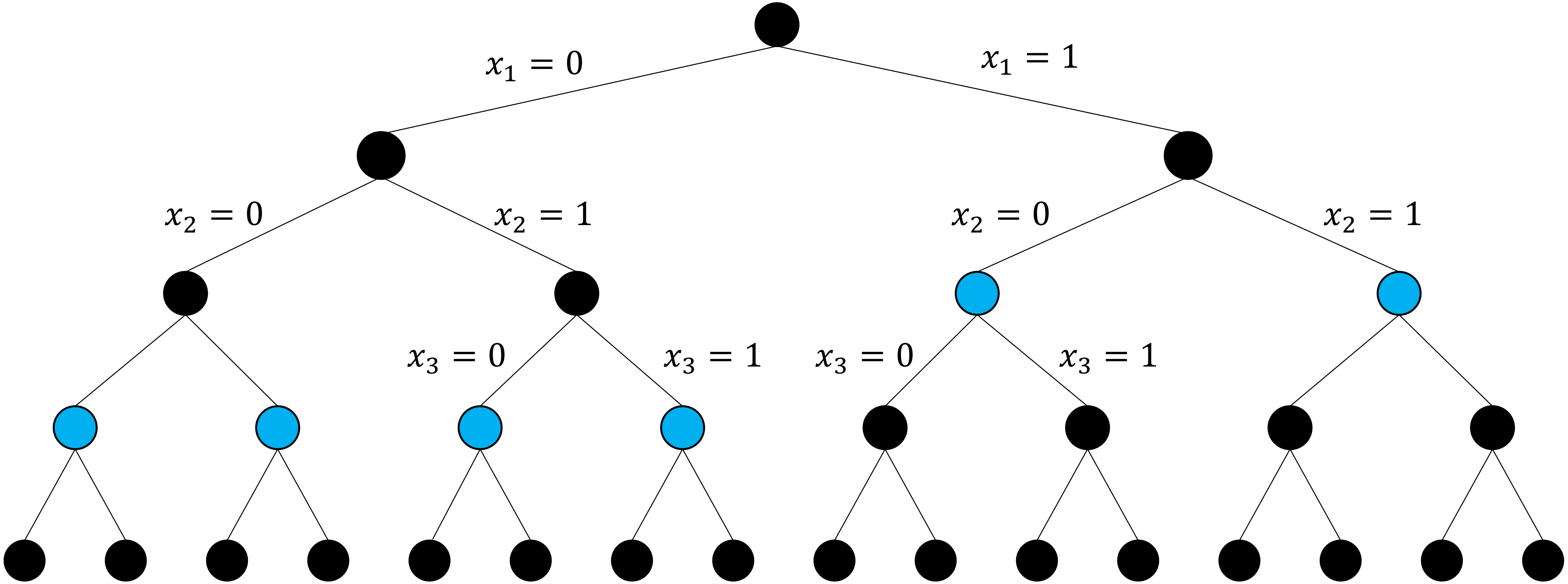
Many different ways to configure/optimize this algorithm, e.g.:

- Node selection policy
- Variable selection policy
- …

What's the best **configuration** for the application at hand?

Best configuration for **routing** problems
likely not suited for **scheduling**

Node selection policy

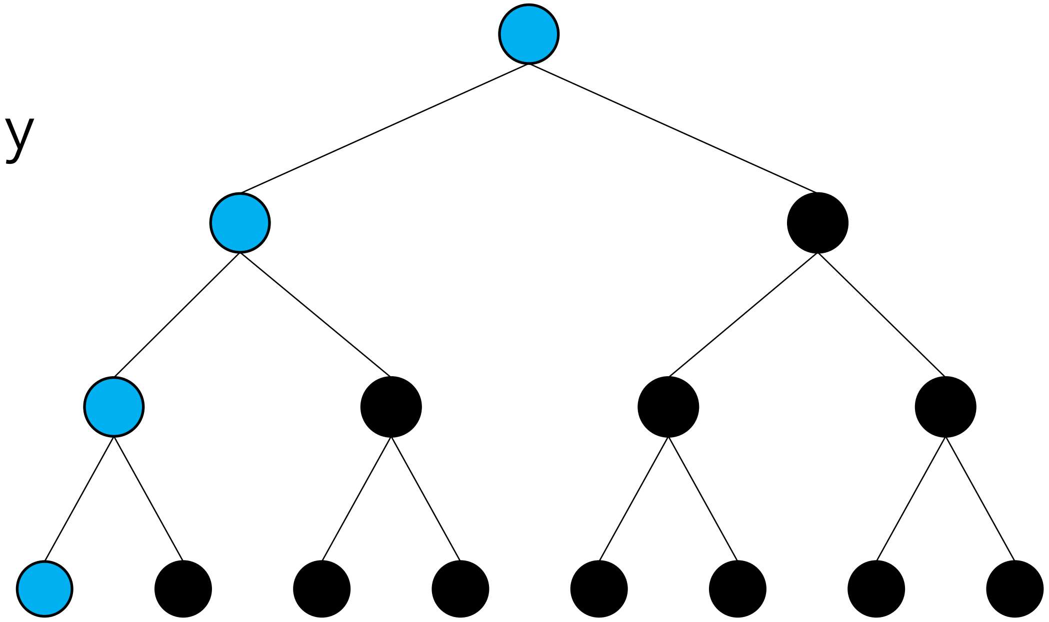


Among many active nodes, which to explore next?

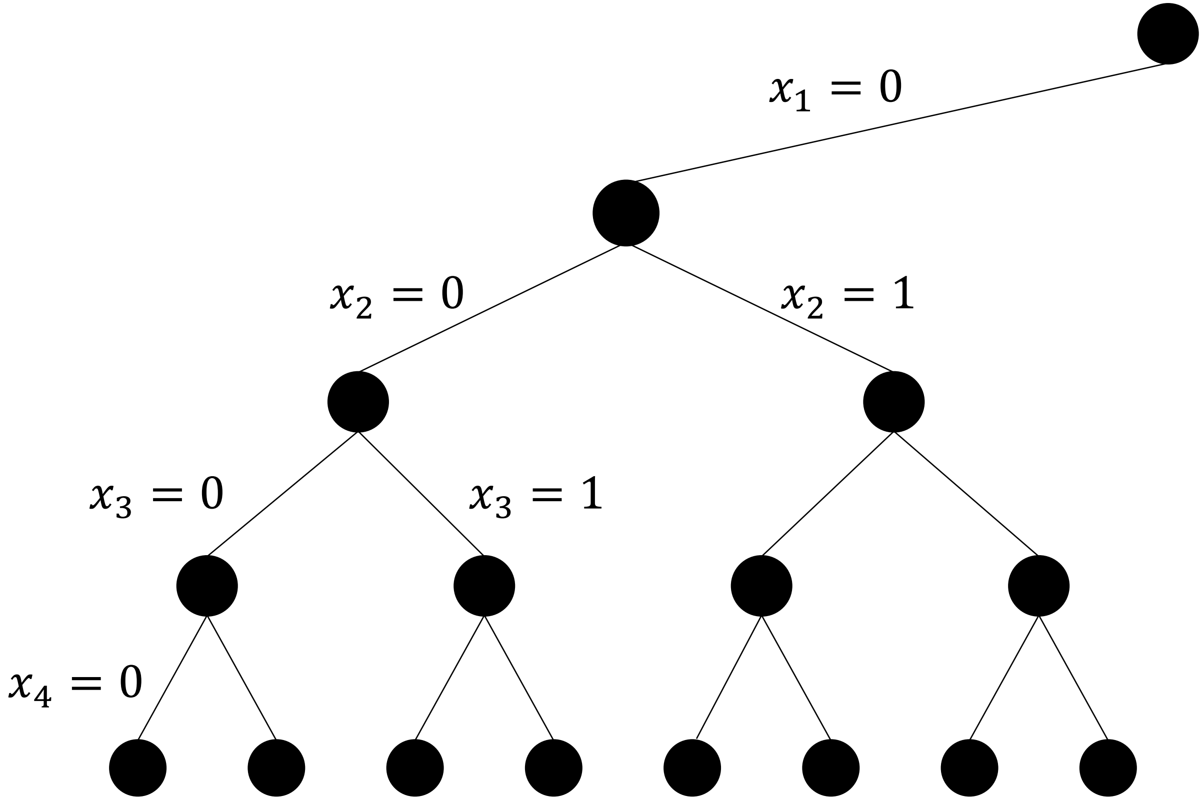
Node selection policy

Among many active nodes, which to explore next?

- *Depth-first search (DFS)*
  - Finds incumbent solutions quickly
- *Best-first search (BFS)*
  - Explore node with highest LP objective value
  - The "most promising" nodes
- *BFS with plunging*
  - Mix of BFS and DFS

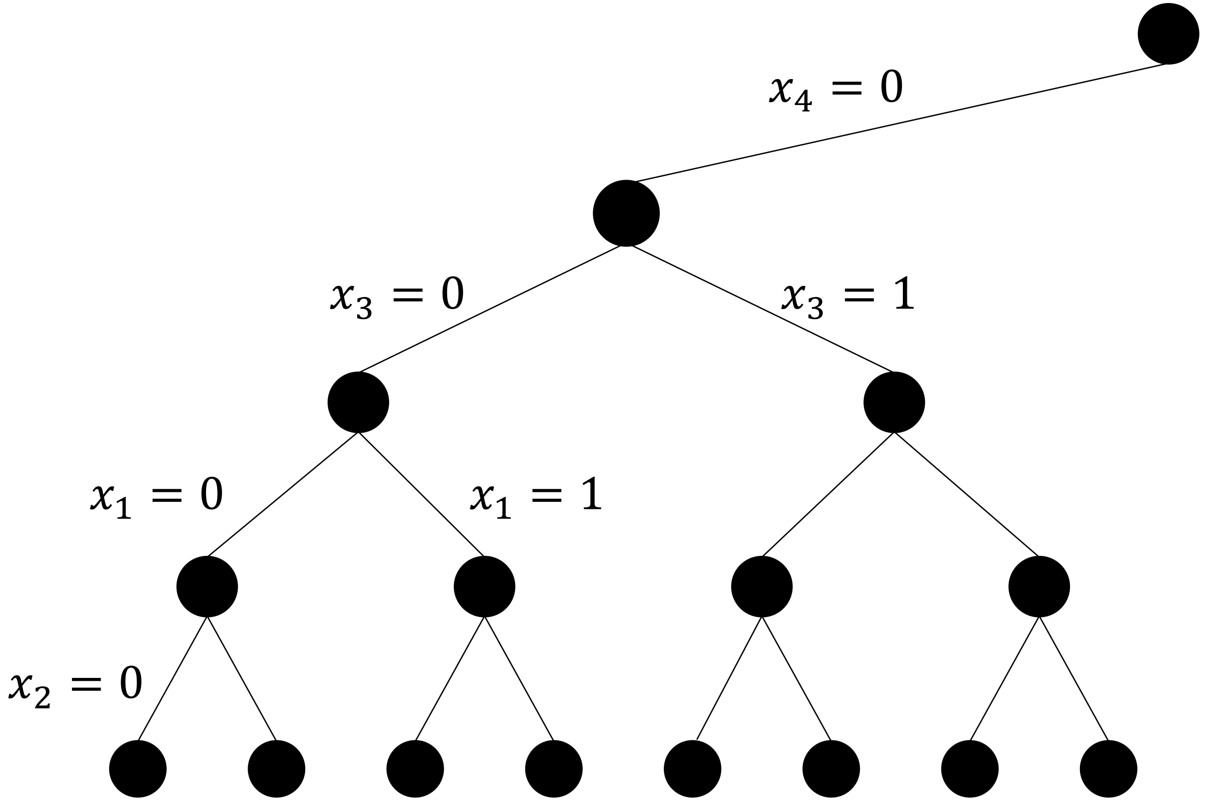


Variable selection policy



Better branching order than x_1, x_2, x_3, x_4?

Variable selection policy

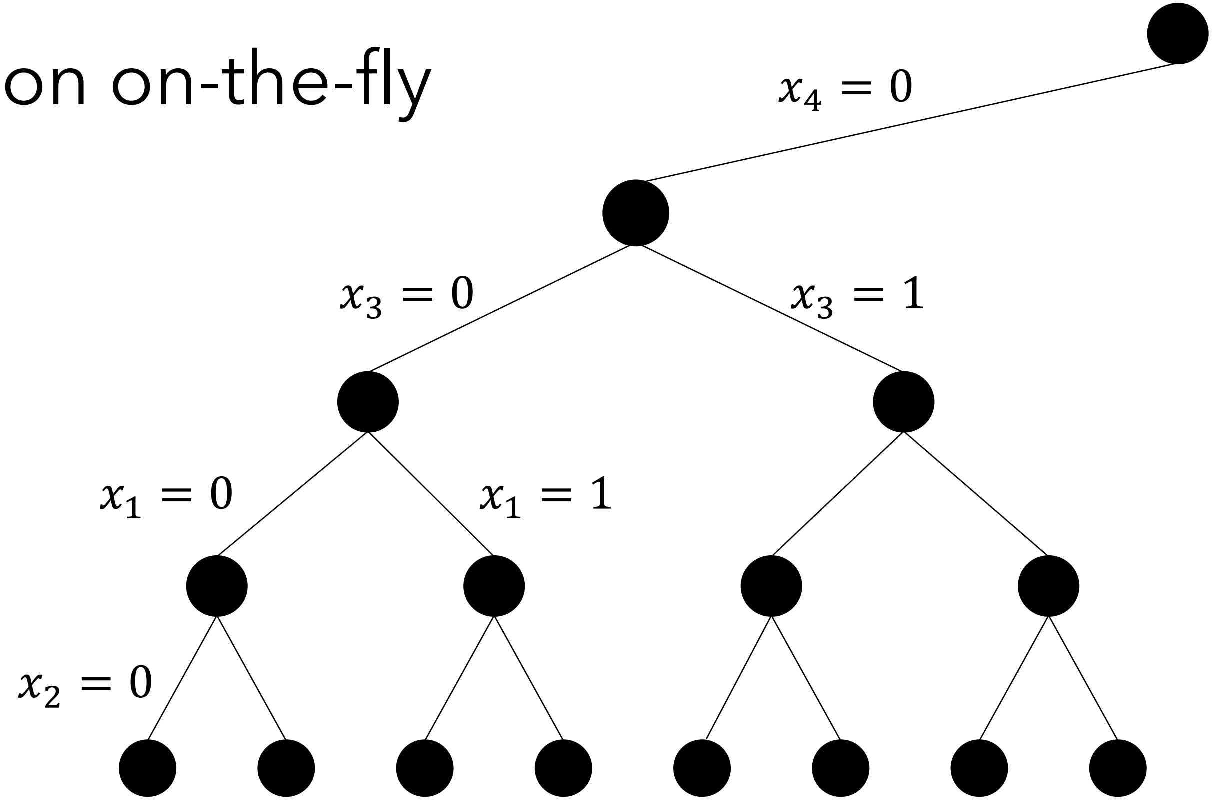


Better branching order than x_1, x_2, x_3, x_4? E.g., x_4, x_3, x_1, x_2

Variable selection policy

Chooses variables to branch on on-the-fly

Rather than pre-defined order



Variable selection policy

On Problem(j) with LP objective value $z(j)$:

- Let $z_i^+(j)$ be the LP objective value after setting $x_i = 1$
- Let $z_i^-(j)$ be the LP objective value after setting $x_i = 0$
- Branch on the variable x_i that maximizes

$$\max\{z(j) - z_i^+(j), z(j) - z_i^-(j)\}$$

Maximal change in objective value

Variable selection policy

On Problem(j) with LP objective value $z(j)$:

- Let $z_i^+(j)$ be the LP objective value after setting $x_i = 1$
- Let $z_i^-(j)$ be the LP objective value after setting $x_i = 0$
- Branch on the variable x_i that maximizes

$$\min\{z(j) - z_i^+(j), z(j) - z_i^-(j)\}$$

Minimal change in objective value

Variable selection policy

On Problem(j) with LP objective value $z(j)$:

- Let $z_i^+(j)$ be the LP objective value after setting $x_i = 1$
- Let $z_i^-(j)$ be the LP objective value after setting $x_i = 0$
- Branch on the variable x_i that maximizes

$$\mu \cdot \min\{z(j) - z_i^+(j), z(j) - z_i^-(j)\} + (1 - \mu) \cdot \max\{z(j) - z_i^+(j), z(j) - z_i^-(j)\}$$

For some IPs, it's better to choose μ closer to 0…

For others, it's better to choose μ closer to 1

Often, $\mu = \frac{5}{6}$ works well [Achterberg, '09]

Variable selection policy

On Problem(j) with LP objective value $z(j)$:

- Let $z_i^+(j)$ be the LP objective value after setting $x_i = 1$
- Let $z_i^-(j)$ be the LP objective value after setting $x_i = 0$
- Branch on the variable x_i that maximizes

$$\mu \cdot \min\{z(j) - z_i^+(j), z(j) - z_i^-(j)\} + (1 - \mu) \cdot \max\{z(j) - z_i^+(j), z(j) - z_i^-(j)\}$$

Challenge: Computing $z_i^-(j), z_i^+(j)$ requires solving a lot of LPs

Computing all of these LP relaxations referred to as "*strong-branching*"

Pseudo-cost branching: only use estimates of these LP values

Variable selection policy

On Problem(j) with LP objective value $z(j)$:

- Let $z_i^+(j)$ be the LP objective value after setting $x_i = 1$
- Let $z_i^-(j)$ be the LP objective value after setting $x_i = 0$
- Branch on the variable x_i that maximizes

$$\mu \cdot \min\{z(j) - z_i^+(j), z(j) - z_i^-(j)\} + (1 - \mu) \cdot \max\{z(j) - z_i^+(j), z(j) - z_i^-(j)\}$$

Many other possible variable selection policies!

See, e.g., [Achterberg, '09]

Major challenge of using B&B

How to choose the best {*node*, *variable*, …}-selection policy?

Little theory about which to use when

This course:

Use machine learning to optimize B&B's performance

Outline

1. Linear programming
2. Integer programming
3. **SAT solving** (SAT solvers also use tree search)
4. Next steps

SAT refresher

- $x_1, x_2, x_3, \ldots \in \{0,1\}$
- $\bar{x}_1$ means **Not** x_1
  - If $x_1 = 1$ then $\bar{x}_1 = 0$; if $x_1 = 0$ then $\bar{x}_1 = 1$
- $\lor$ means **Or**
  - $x_1 \lor x_2$ evaluates to **True** if $x_1 = 1$ **or** $x_2 = 1$
  - $x_1 \lor \bar{x}_2$ evaluates to **True** if $x_1 = 1$ **or** $x_2 = 0$
- $\land$ means **And**
  - $x_1 \land x_2$ evaluates to **True** if $x_1 = 1$ **and** $x_2 = 1$
  - $x_1 \land \bar{x}_2$ evaluates to **True** if $x_1 = 1$ **and** $x_2 = 0$
- $(x_1 \lor x_2) \land (x_2 \lor \bar{x}_3)$ evaluates to **True** for $(x_1, x_2, x_3) = (1,0,0)$

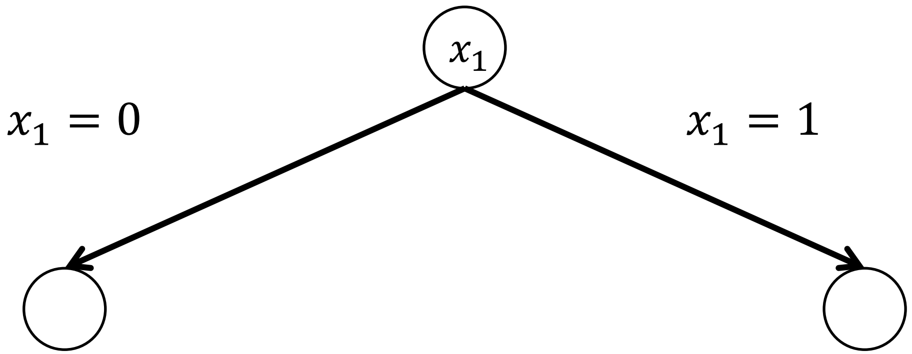
SAT refresher

$$
\begin{aligned}
&(x_1 \lor x_4) \\
\land\ &(x_1 \lor \bar{x}_3 \lor \bar{x}_8) \\
\land\ &(x_1 \lor x_8 \lor x_{12}) \\
\land\ &(x_2 \lor x_{11}) \\
\land\ &(\bar{x}_7 \lor \bar{x}_3 \lor x_9) \\
\land\ &(\bar{x}_7 \lor x_8 \lor \bar{x}_9) \\
\land\ &(x_7 \lor x_8 \lor \bar{x}_{10}) \\
\land\ &(x_7 \lor x_{10} \lor \bar{x}_{12})
\end{aligned}
$$

SAT: Is there an assignment of $x_1, \ldots, x_{12} \in \{0,1\}$ such that this formula evaluates to **True**?

SAT tree search

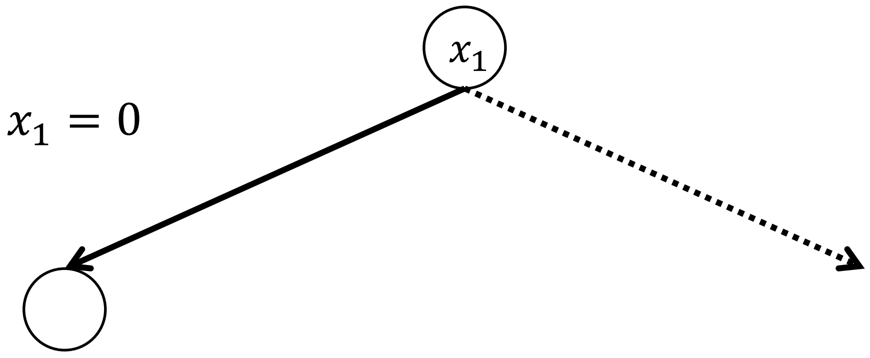
$$
\begin{aligned}
&(x_1 \vee x_4) \\
\wedge\ &(x_1 \vee \bar{x}_3 \vee \bar{x}_8) \\
\wedge\ &(x_1 \vee x_8 \vee x_{12}) \\
\wedge\ &(x_2 \vee x_{11}) \\
\wedge\ &(\bar{x}_7 \vee \bar{x}_3 \vee x_9) \\
\wedge\ &(\bar{x}_7 \vee x_8 \vee \bar{x}_9) \\
\wedge\ &(x_7 \vee x_8 \vee \bar{x}_{10}) \\
\wedge\ &(x_7 \vee x_{10} \vee \bar{x}_{12})
\end{aligned}
$$



How to prune if there's no objective function?

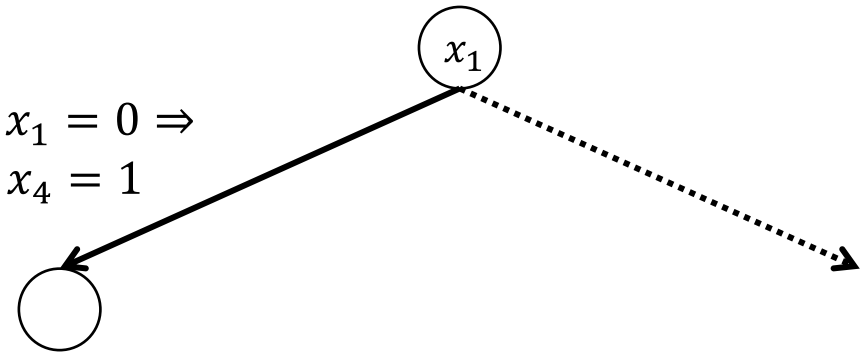
SAT tree search

$$
\begin{aligned}
& (x_1 \vee x_4) \\
\wedge\ & (x_1 \vee \bar{x}_3 \vee \bar{x}_8) \\
\wedge\ & (x_1 \vee x_8 \vee x_{12}) \\
\wedge\ & (x_2 \vee x_{11}) \\
\wedge\ & (\bar{x}_7 \vee \bar{x}_3 \vee x_9) \\
\wedge\ & (\bar{x}_7 \vee x_8 \vee \bar{x}_9) \\
\wedge\ & (x_7 \vee x_8 \vee \bar{x}_{10}) \\
\wedge\ & (x_7 \vee x_{10} \vee \bar{x}_{12})
\end{aligned}
$$



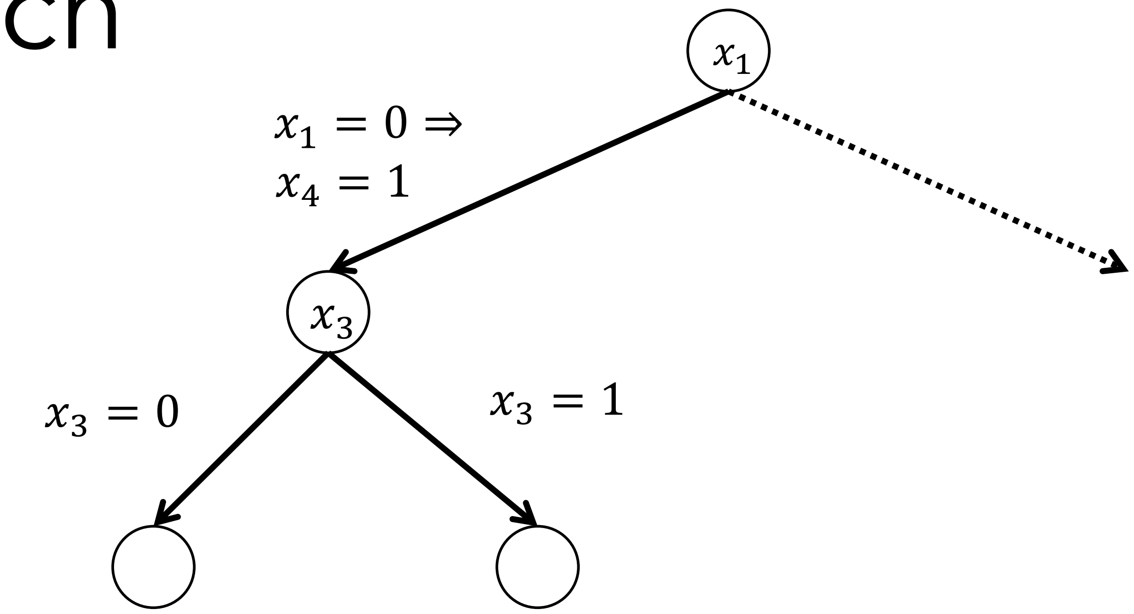
SAT tree search

$$
\begin{aligned}
&(x_1 \lor x_4) \\
\land\, &(x_1 \lor \bar{x}_3 \lor \bar{x}_8) \\
\land\, &(x_1 \lor x_8 \lor x_{12}) \\
\land\, &(x_2 \lor x_{11}) \\
\land\, &(\bar{x}_7 \lor \bar{x}_3 \lor x_9) \\
\land\, &(\bar{x}_7 \lor x_8 \lor \bar{x}_9) \\
\land\, &(x_7 \lor x_8 \lor \bar{x}_{10}) \\
\land\, &(x_7 \lor x_{10} \lor \bar{x}_{12})
\end{aligned}
$$



SAT tree search

$$
\begin{aligned}
&(x_1 \vee x_4)\\
\wedge\ &(x_1 \vee \bar{x}_3 \vee \bar{x}_8)\\
\wedge\ &(x_1 \vee x_8 \vee x_{12})\\
\wedge\ &(x_2 \vee x_{11})\\
\wedge\ &(\bar{x}_7 \vee \bar{x}_3 \vee x_9)\\
\wedge\ &(\bar{x}_7 \vee x_8 \vee \bar{x}_9)\\
\wedge\ &(x_7 \vee x_8 \vee \bar{x}_{10})\\
\wedge\ &(x_7 \vee x_{10} \vee \bar{x}_{12})
\end{aligned}
$$



SAT tree search

$$
\begin{aligned}
& (x_1 \lor x_4) \\
\land\ & (x_1 \lor \bar{x}_3 \lor \bar{x}_8) \\
\land\ & (x_1 \lor x_8 \lor x_{12}) \\
\land\ & (x_2 \lor x_{11}) \\
\land\ & (\bar{x}_7 \lor \bar{x}_3 \lor x_9) \\
\land\ & (\bar{x}_7 \lor x_8 \lor \bar{x}_9) \\
\land\ & (x_7 \lor x_8 \lor \bar{x}_{10}) \\
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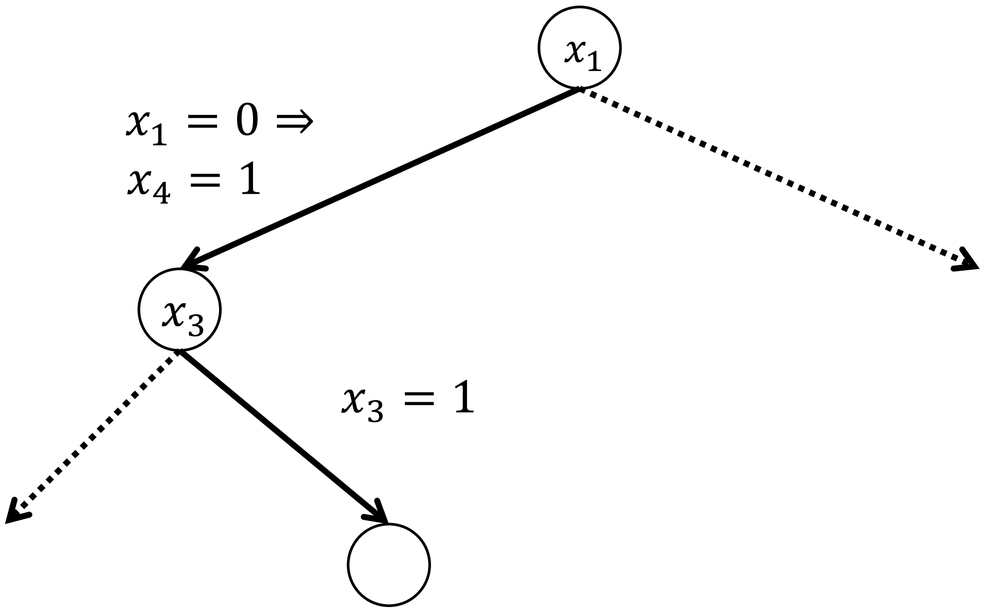
\end{aligned}
$$



SAT tree search

$$
\begin{aligned}
& (x_1 \lor x_4) \\
\land\ & (x_1 \lor \bar{x}_3 \lor \bar{x}_8) \\
\land\ & (x_1 \lor x_8 \lor x_{12}) \\
\land\ & (x_2 \lor x_{11}) \\
\land\ & (\bar{x}_7 \lor \bar{x}_3 \lor x_9) \\
\land\ & (\bar{x}_7 \lor x_8 \lor \bar{x}_9) \\
\land\ & (x_7 \lor x_8 \lor \bar{x}_{10}) \\
\land\ & (x_7 \lor x_{10} \lor \bar{x}_{12})
\end{aligned}
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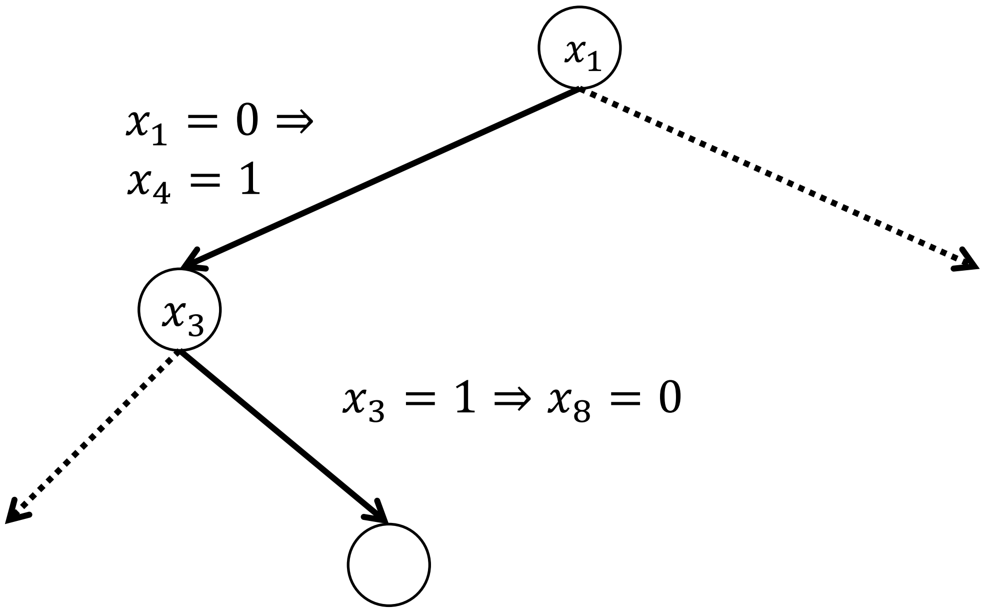
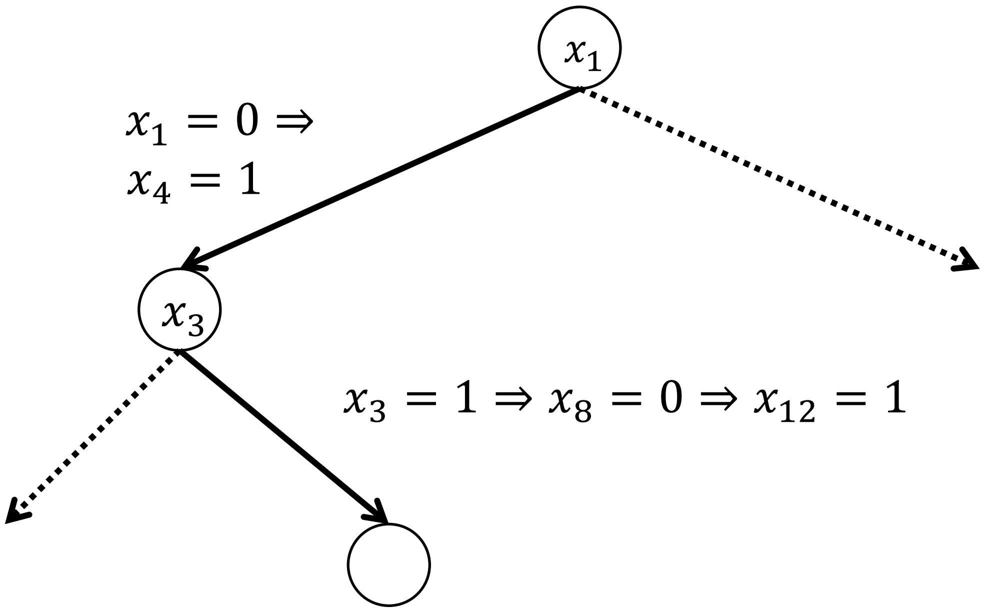


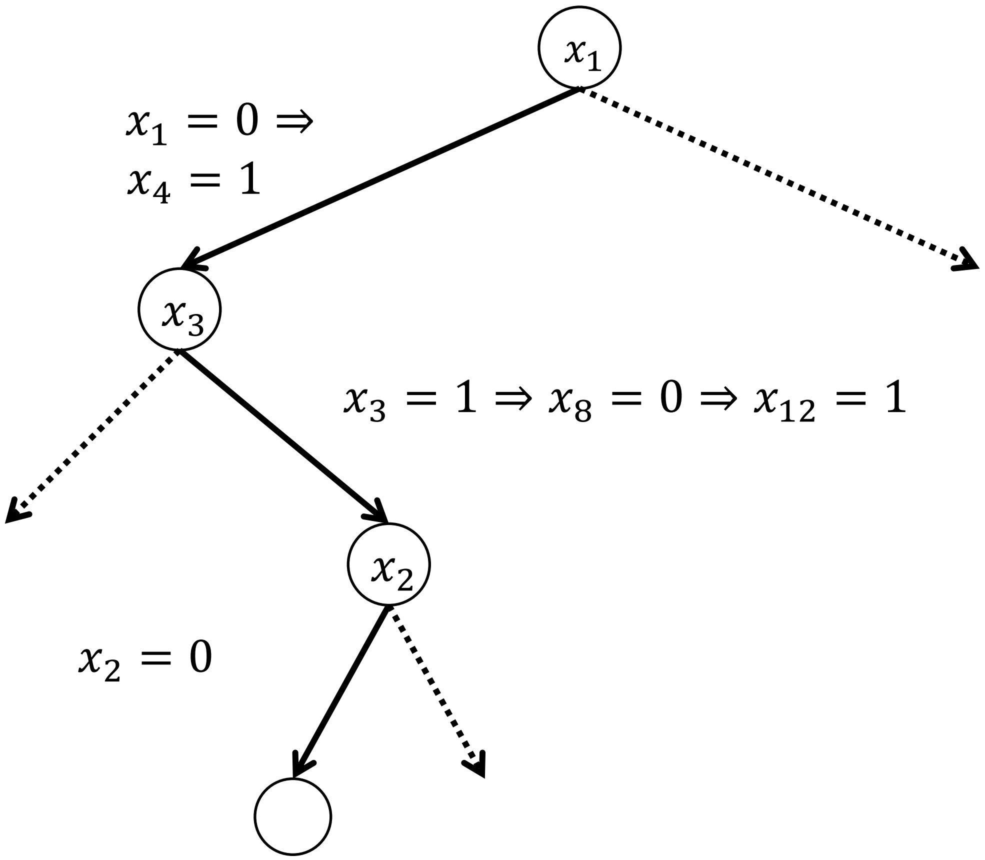
SAT tree search

$$
\begin{aligned}
&(x_1 \vee x_4)\\
\wedge\ &(x_1 \vee \bar{x}_3 \vee \bar{x}_8)\\
\wedge\ &(x_1 \vee x_8 \vee x_{12})\\
\wedge\ &(x_2 \vee x_{11})\\
\wedge\ &(\bar{x}_7 \vee \bar{x}_3 \vee x_9)\\
\wedge\ &(\bar{x}_7 \vee x_8 \vee \bar{x}_9)\\
\wedge\ &(x_7 \vee x_8 \vee \bar{x}_{10})\\
\wedge\ &(x_7 \vee x_{10} \vee \bar{x}_{12})
\end{aligned}
$$



SAT tree search

$$
\begin{aligned}
&(x_1 \lor x_4) \\
\land\ &(x_1 \lor \bar{x}_3 \lor \bar{x}_8) \\
\land\ &(x_1 \lor x_8 \lor x_{12}) \\
\land\ &(x_2 \lor x_{11}) \\
\land\ &(\bar{x}_7 \lor \bar{x}_3 \lor x_9) \\
\land\ &(\bar{x}_7 \lor x_8 \lor \bar{x}_9) \\
\land\ &(x_7 \lor x_8 \lor \bar{x}_{10}) \\
\land\ &(x_7 \lor x_{10} \lor \bar{x}_{12})
\end{aligned}
$$



SAT tree search

$$
\begin{aligned}
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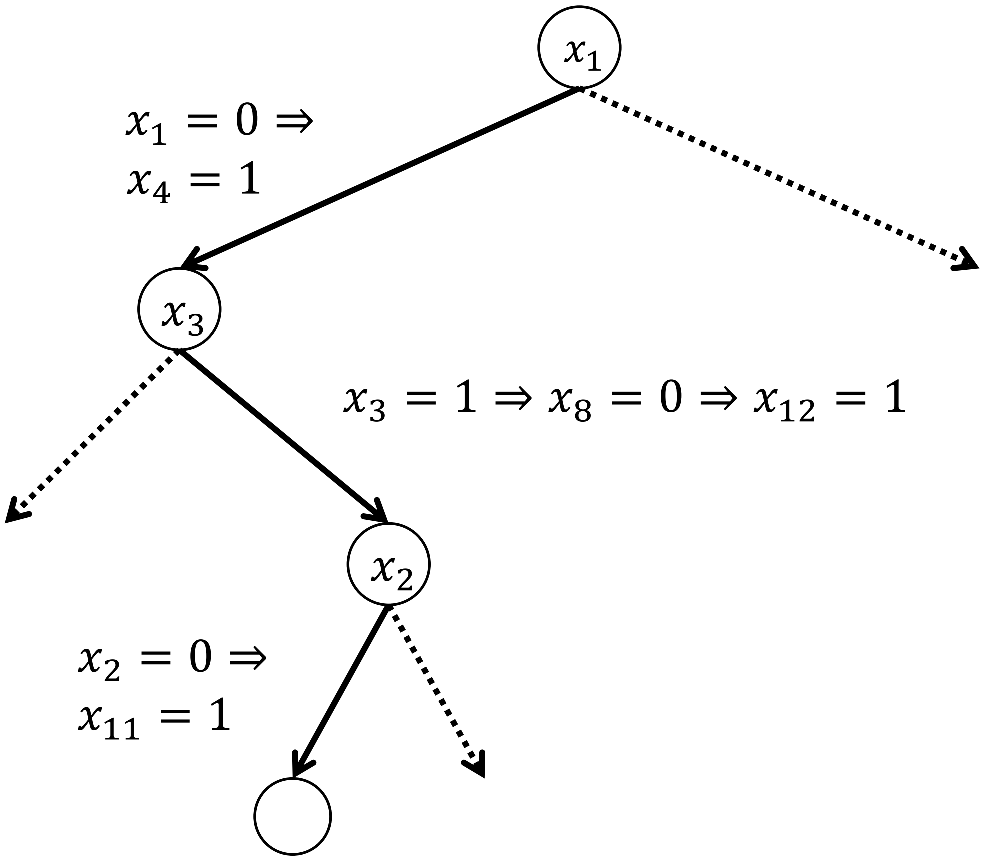
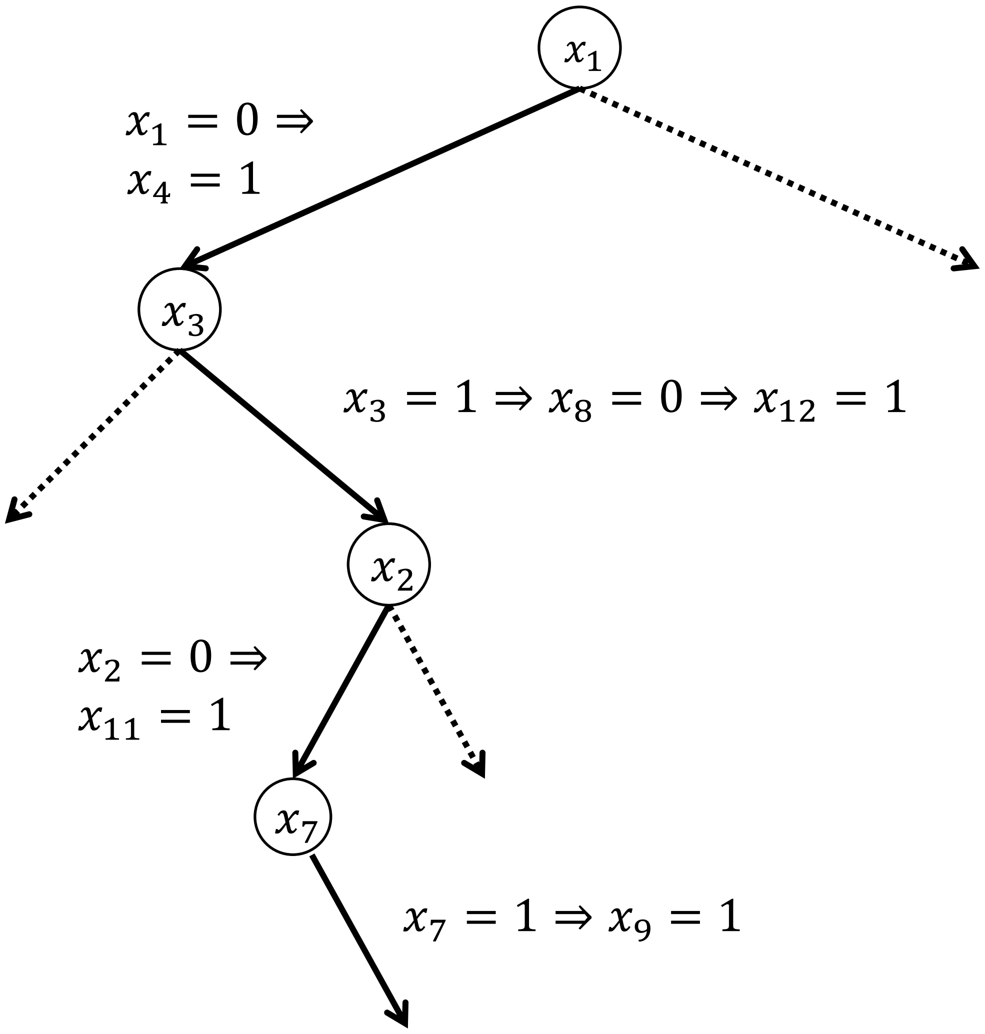
\wedge\ & (x_1 \vee \bar{x}_3 \vee \bar{x}_8) \\
\wedge\ & (x_1 \vee x_8 \vee x_{12}) \\
\wedge\ & (x_2 \vee x_{11}) \\
\wedge\ & (\bar{x}_7 \vee \bar{x}_3 \vee x_9) \\
\wedge\ & (\bar{x}_7 \vee x_8 \vee \bar{x}_9) \\
\wedge\ & (x_7 \vee x_8 \vee \bar{x}_{10}) \\
\wedge\ & (x_7 \vee x_{10} \vee \bar{x}_{12})
\end{aligned}
$$



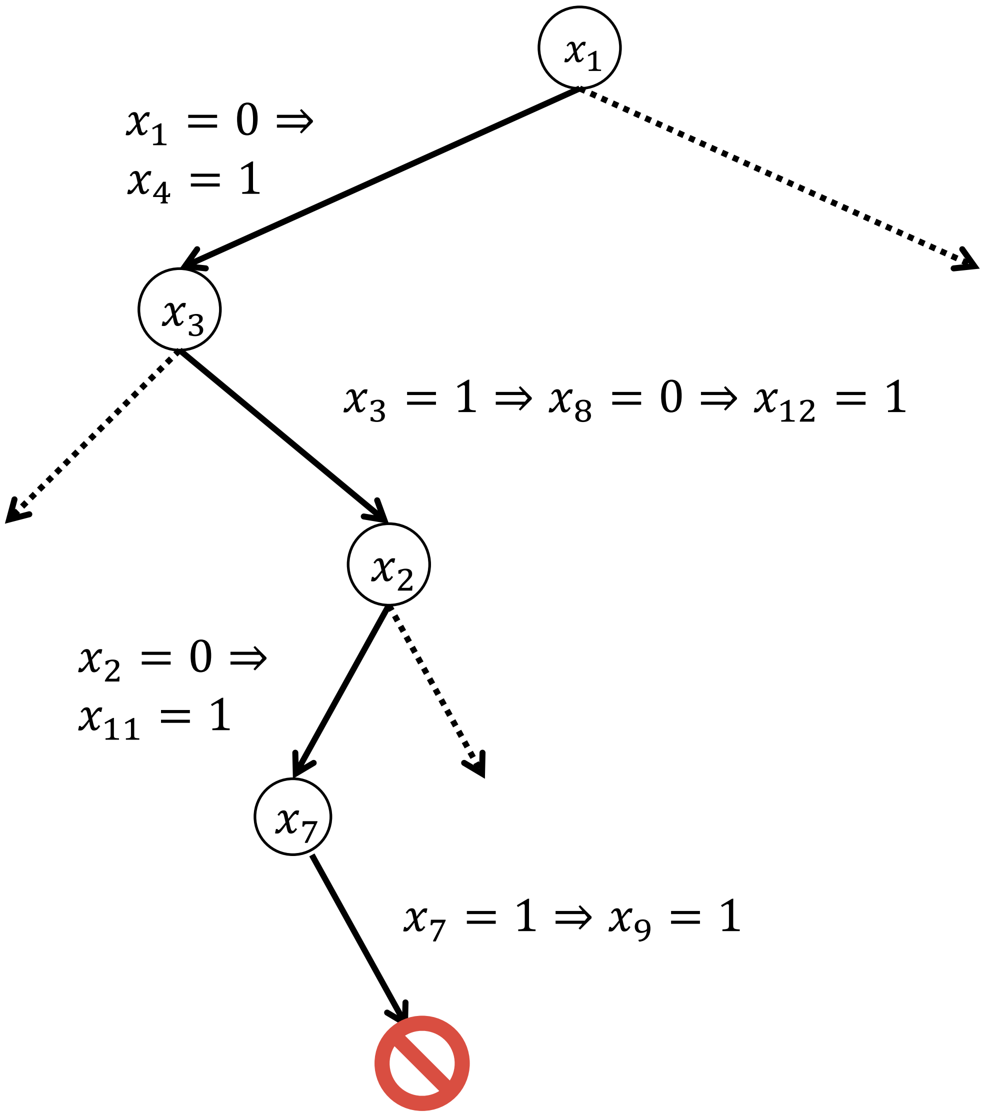
SAT tree search

$$
\begin{aligned}
&(x_1 \lor x_4) \\
\land\ &(x_1 \lor \bar{x}_3 \lor \bar{x}_8) \\
\land\ &(x_1 \lor x_8 \lor x_{12}) \\
\land\ &(x_2 \lor x_{11}) \\
\land\ &(\bar{x}_7 \lor \bar{x}_3 \lor x_9) \\
\land\ &(\bar{x}_7 \lor x_8 \lor \bar{x}_9) \\
\land\ &(x_7 \lor x_8 \lor \bar{x}_{10}) \\
\land\ &(x_7 \lor x_{10} \lor \bar{x}_{12})
\end{aligned}
$$



SAT tree search

$$
\begin{aligned}
&(x_1 \lor x_4) \\
\land\ &(x_1 \lor \bar{x}_3 \lor \bar{x}_8) \\
\land\ &(x_1 \lor x_8 \lor x_{12}) \\
\land\ &(x_2 \lor x_{11}) \\
\land\ &(\bar{x}_7 \lor \bar{x}_3 \lor x_9) \\
\land\ &(\bar{x}_7 \lor x_8 \lor \bar{x}_9) \\
\land\ &(x_7 \lor x_8 \lor \bar{x}_{10}) \\
\land\ &(x_7 \lor x_{10} \lor \bar{x}_{12})
\end{aligned}
$$



SAT tree search

Many similar design choices as IP tree search

Variable selection: branch on variable that leads to the **largest number of deductions** on other variables

Outline

1. Linear programming
2. Integer programming
3. SAT solving
4. **Next steps**

Next steps

- **Today:** Saw many ways solvers can be optimized & configured
- With a deft configuration:
  Can quickly solve extremely challenging real-world problems

Routing

Manufacturing

Scheduling

Planning

Finance

- **Future classes:** How to use ML to optimize these solvers

Plan for the next few classes

This Thursday: GNNs overview

Tu 4/18, Th 4/20: GNN paper discussions

Tu 4/25, Th 4/27: IP/SAT paper discussions